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Subject Code :EC8451	**Subject Name** :ELECTROMAGNETIC FIELDS	Rev. No.	02
Unit No :1	**Unit Name** : INTRODUCTION	Date	14-11-2017

LECTURE NOTES

Introduction:

Electromagnetic theory is a discipline concerned with the study of charges at rest and in motion. Electromagnetic principles are fundamental to the study of electrical engineering and physics. Electromagnetic theory is also indispensable to the understanding, analysis and design of various electrical, electromechanical and electronic systems. Some of the branches of study where electromagnetic principles find application are:

- RF communication
- Microwave Engineering
- Antennas
- Electrical Machines
- Satellite Communication
- Atomic and nuclear research
- Radar Technology
- Remote sensing
- EMI EMC
- Quantum Electronics
- VLSI

Electromagnetic theory is a prerequisite for a wide spectrum of studies in the field of Electrical Sciences and Physics. Electromagnetic theory can be thought of as generalization of circuit theory. There are certain situations that can be handled exclusively in terms of field theory. In electromagnetic theory, the quantities involved can be categorized as **source quantities** and **field quantities**. Source of electromagnetic field is electric charges: either at rest or in motion. However an electromagnetic field may cause a redistribution of charges that in turn change the field and hence the separation of cause and effect is not always visible.

Electric charge is a fundamental property of matter. Charge exist only in positive or negative integral multiple of **electronic charge**, -e, e= 1.60×10^{-19} coulombs. [It may be noted here that in 1962, Murray Gell-Mann hypothesized **Quarks** as the basic building blocks of matters. Quarks were predicted to carry a fraction of electronic charge and the existence of Quarks has been experimentally verified.] Principle of conservation of charge states that the total charge (algebraic sum of positive and negative charges) of an isolated system remains unchanged, though the charges may redistribute under the influence of electric field. Kirchhoff's Current Law (KCL) is an assertion of the conservative property of charges under the implicit assumption that there is no accumulation of charge at the junction.

Electromagnetic theory deals directly with the electric and magnetic field vectors whereas circuit theory deals with the voltages and currents. Voltages and currents are integrated effects of electric and magnetic fields respectively. Electromagnetic field problems involve three space variables along with the time variable and hence the solution tends to become correspondingly complex. Vector analysis is a mathematical tool with which electromagnetic concepts are more conveniently expressed and best comprehended. Since use of vector analysis in the study of electromagnetic field theory results in real economy of time and thought, we first introduce the concept of vector analysis.

Vector Analysis:

The quantities that we deal in electromagnetic theory may be either **scalar** or **vectors** [There are other classes of physical quantities called **Tensors**: where magnitude and direction vary with co-ordinate axes]. Scalars are quantities characterized by magnitude only and algebraic sign. A quantity that has direction as well as magnitude is called a vector. Both scalar and vector quantities are function of *time* and *position*. A field is a function that specifies a particular quantity everywhere in a region. Depending upon the nature of the quantity under consideration, the field may be a vector or a scalar field. Example of scalar field is the electric potential in a region while electric or magnetic fields at any point is the example of vector field.

A vector $\vec{A}$ can be written as, $\vec{A} = \hat{a} A$, where, $A = |\vec{A}|$ is the magnitude and $\hat{a} = \frac{\vec{A}}{|A|}$ is the unit vector which has unit magnitude and same direction as that of $\vec{A}$.

Two vector $\vec{A}$ and $\vec{B}$ are added together to give another vector $\vec{C}$. We have

$$\vec{C} = \vec{A} + \vec{B} \quad ....(1.1)$$

Let us see the animations in the next pages for the addition of two vectors, which has two rules:

1: Parallelogram law and **2: Head & tail rule**

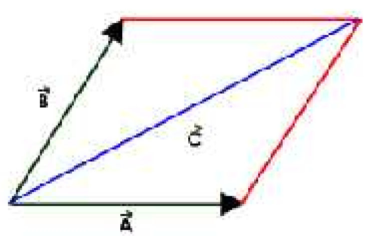


Fig 1.1(a):Vector Addition(Parallelogram Rule)

LECTURE NOTES

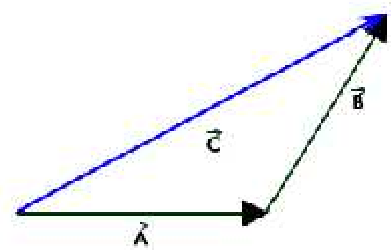


Fig 1.1(b): Vector Addition (Head & Tail Rule)

Vector Subtraction is similarly carried out: $\vec{D} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ ....(1.2)

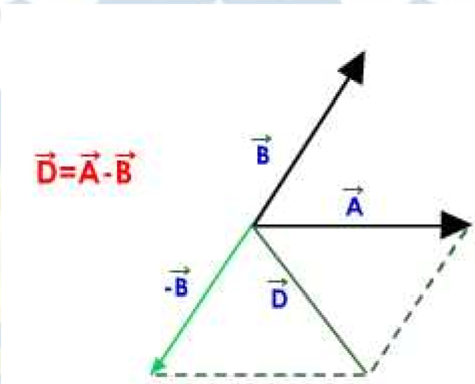


Fig 1.2: Vector subtraction

Scaling of a vector is defined as $\vec{C} = \alpha\vec{B}$, where $\vec{C}$ is scaled version of vector $\vec{B}$ and α is a scalar.
Some important laws of vector algebra are:

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$ Commutative Law..(1.3)

$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$ Associative Law...(1.4)

$\alpha(\vec{A} + \vec{B}) = \alpha\vec{A} + \alpha\vec{B}$ Distributive Law ....(1.5)

The position vector $\vec{r_P}$ of a point *P* is the directed distance from the origin (*O*) to *P*, i.e., $\vec{r_P} = \overrightarrow{OP}$

If $\vec{r_Q}$ = OP and $\vec{r_P}$ = OQ are the position vectors of the points P and Q then the distance vector

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r_P} - \vec{r_Q}$$

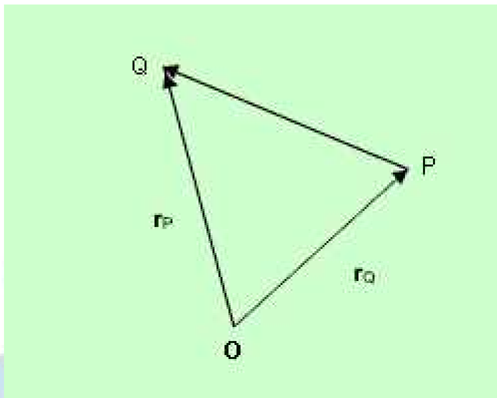


Fig 1.3: Distance Vector

Product of Vectors

When two vectors $\vec{A}$ and $\vec{B}$ are multiplied, the result is either a scalar or a vector depending how the two vectors were multiplied. The two types of vector multiplication are:

Scalar product (or dot product) $\vec{A}\cdot\vec{B}$ gives a scalar.

Vector product (or cross product) $\vec{A}\times\vec{B}$ gives a vector.

The dot product between two vectors is defined as $\vec{A}\cdot\vec{B} = |A||B|cos\theta_{AB}$ ....(1.6)

Vector product $\vec{A}\times\vec{B} = |A||B|\sin\theta_{AB}\cdot\vec{n}$

$\vec{n}$ is unit vector perpendicular to $\vec{A}$ and $\vec{B}$

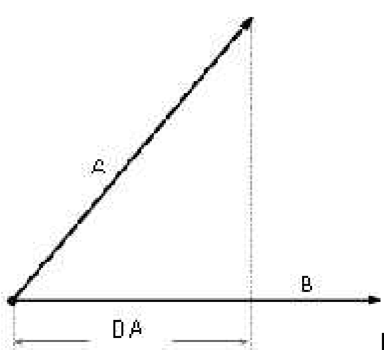


Fig 1.4: Vector Dot Product

The dot product is commutative i.e., $\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$ and distributive i.e., $\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$. Associative law does not apply to scalar product.

The vector or cross product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\times\vec{B}$. $\vec{A}\times\vec{B}$ is a vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, the magnitude is given by $|A||B|\sin\theta_{AB}$ and direction is given by right hand rule as explained in Figure 1.5.

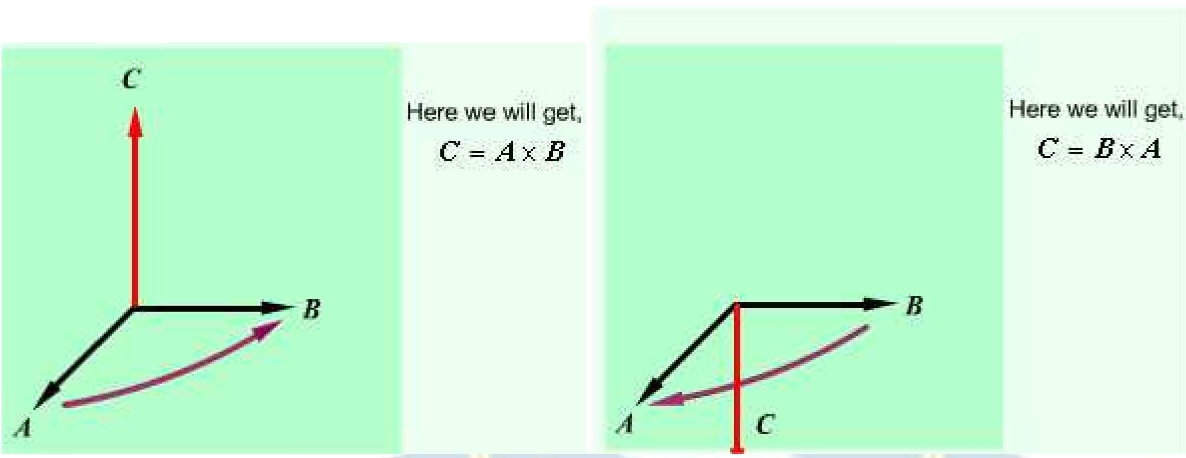


Fig 1.5 :Illustrating the left thumb rule for determining the vector cross product

$\vec{A}\times\vec{B}=\hat{a}_n\,AB\sin\theta_{AB}$ ....(1.7)

where $\hat{a}_n$ is the unit vector given by, $\hat{a}_n=\dfrac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|}$.

The following relations hold for vector product.

$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$ i.e., cross product is non commutative ....(1.8)

$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$ i.e., cross product is distributive.......................(1.9)

$\vec{A}\times\left(\vec{B}\times\vec{C}\right)\neq\left(\vec{A}\times\vec{B}\right)\times\vec{C}$ i.e., cross product is non associative..............(1.10)

Scalar and vector triple product :

Scalar triple product $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)=\vec{C}\cdot\left(\vec{A}\times\vec{B}\right)$ ....(1.11)

Vector triple product $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)$..................................(1.12)

CO-ORDINATE SYSTEMS

In order to describe the spatial variations of the quantities, we require using appropriate co-ordinate system. A point or vector can be represented in a **curvilinear** coordinate system that may be **orthogonal** or **non-orthogonal**.

An orthogonal system is one in which the co-ordinates are mutually perpendicular. Non-orthogonal co-ordinate systems are also possible, but their usage is very limited in practice.

Let u = constant, v = constant and w = constant represent surfaces in a coordinate system, the surfaces may be curved surfaces in general. Further, let $\hat{a}_u$, $\hat{a}_v$ and $\hat{a}_w$ be the unit vectors in the three coordinate directions(base vectors). In general right handed orthogonal curvilinear systems, the vectors satisfy the following relations:

$$\hat{a}_u\times\hat{a}_v=\hat{a}_w$$
$$\hat{a}_v\times\hat{a}_w=\hat{a}_u$$
$$\hat{a}_w\times\hat{a}_u=\hat{a}_v \quad \text{....................................(1.13)}$$

These equations are not independent and specification of one will automatically imply the other two. Furthermore, the following relations hold.

$$\hat{a}_u\cdot\hat{a}_v=\hat{a}_v\cdot\hat{a}_w=\hat{a}_w\cdot\hat{a}_u=0$$
$$\hat{a}_u\cdot\hat{a}_u=\hat{a}_v\cdot\hat{a}_v=\hat{a}_w\cdot\hat{a}_w=1 \quad \text{................(1.14)}$$

A vector can be represented as sum of its orthogonal components, $\vec{A}=A_u\hat{a}_u+A_v\hat{a}_v+A_w\hat{a}_w$ ....(1.15)
In general u, v and w may not represent length. We multiply u, v and w by conversion factors h_1,h_2 and h_3 respectively to convert differential changes du, dv and dw to corresponding changes in length dl_1, dl_2, and dl_3. Therefore

$$d\vec{l}=\hat{a}_u\,dl_1+\hat{a}_v\,dl_2+\hat{a}_w\,dl_3$$
$$=h_1du\,\hat{a}_u+h_2dv\,\hat{a}_v+h_3dw\,\hat{a}_w \quad \text{...............(1.16)}$$

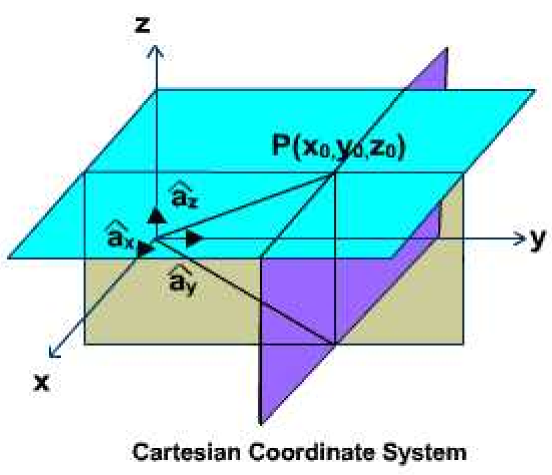
In the same manner, differential volume dv can be written as $dv = h_1 h_2 h_3 du dv dw$ and differential area ds_1 normal to $\hat{a}_u$ is given by, $ds_1 = h_2 h_3 dv dw$. In the same manner, differential areas normal to unit vectors $\hat{a}_v$ and $\hat{a}_w$ can be defined.

In the following sections we discuss three most commonly used orthogonal co-ordinate systems, viz:

1. **Cartesian (or rectangular) co-ordinate system**
2. **Cylindrical co-ordinate system**
3. **Spherical polar co-ordinate system**

Cartesian Co-ordinate System :

In Cartesian co-ordinate system, we have, $(u,v,w) = (x,y,z)$. A point $P(x_0, y_0, z_0)$ in Cartesian co-ordinate system is represented as intersection of three planes $x = x_0$, $y = y_0$ and $z = z_0$. The unit vectors satisfies the following relation:



$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$
$$\hat{a}_y \times \hat{a}_z = \hat{a}_x$$
$$\hat{a}_z \times \hat{a}_x = \hat{a}_y$$
$$\hat{a}_x \cdot \hat{a}_y = \hat{a}_y \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_x = 0$$
$$\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1 \quad \text{....................(1.17)}$$

$$\overrightarrow{OP} = \hat{a}_x x_0 + \hat{a}_y y_0 + \hat{a}_z z_0 \quad \text{....................(1.18)}$$

Fig 1.6:Cartesian co-ordinate system

In cartesian co-ordinate system, a vector $\vec{A}$ can be written as $\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$. The dot and cross product of two vectors $\vec{A}$ and $\vec{B}$ can be written as follows:

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$ ....(1.19)

$$\vec{A}\times\vec{B} = \hat{a}_x(A_yB_z - A_zB_y) + \hat{a}_y(A_zB_x - A_xB_z) + \hat{a}_z(A_xB_y - A_yB_x)$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$ ....(1.20)

Since x, y and z all represent lengths, $h_1 = h_2 = h_3 = 1$. The differential length, area and volume are defined respectively as

$d\vec{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$ ....(1.21)

$d\vec{s}_x = dydz\,\hat{a}_x$

$d\vec{s}_y = dxdz\,\hat{a}_y$

$d\vec{s}_z = dxdy\,\hat{a}_z$

$d\upsilon = dxdydz$ ....(1.22)

Cylindrical Co-ordinate System :

For cylindrical coordinate systems we have $(u, v, w) = (r, \phi, z)$ a point $P(r_0, \phi_0, z_0)$ is determined as the point of intersection of a cylindrical surface $r = r_0$, half plane containing the z-axis and making an angle $\phi = \phi_0$; with the xz plane and a plane parallel to xy plane located at $z=z_0$ as shown in figure 7 on next page.

In cylindrical coordinate system, the unit vectors satisfy the following relations

$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z$

$\hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho$

$\hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi$ ....(1.23)

A vector $\vec{A}$ can be written as , $\vec{A} = A_\rho\,\hat{a}_\rho + A_\phi\,\hat{a}_\phi + A_z\,\hat{a}_z$ ....(1.24)

The differential length is defined as,

$d\vec{l} = \hat{a}_\rho\,d\rho + \rho d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$ $\quad h_1 = 1, h_2 = \rho, h_3 = 1$ ....(1.25)

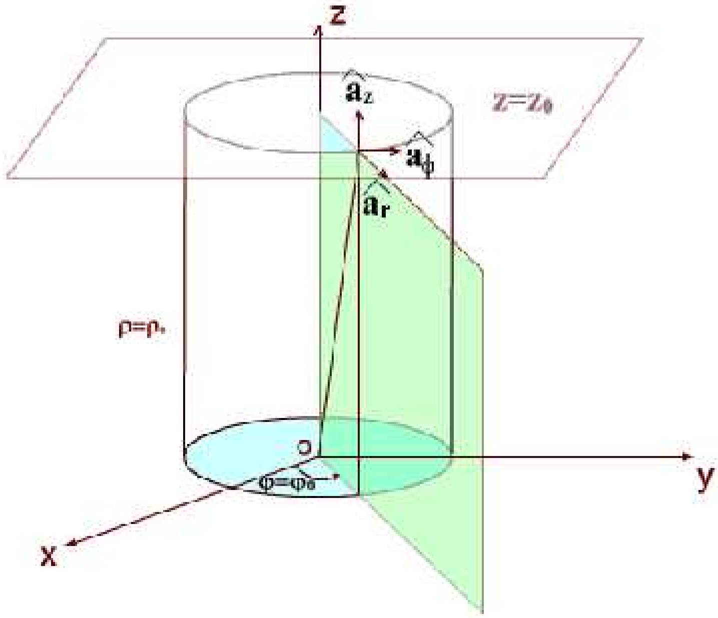
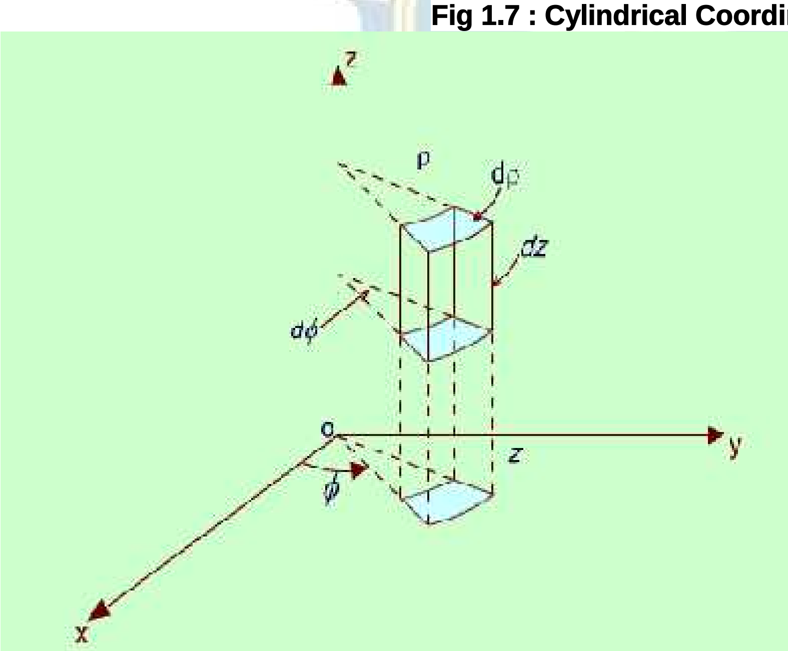


Fig 1.7 : Cylindrical Coordinate System



Differential areas are:

$$\overrightarrow{ds_\rho} = \rho d\phi dz\, \hat{a}_\rho$$

$$\overrightarrow{ds_\phi} = d\rho dz\, \hat{a}_\phi$$

$$\overrightarrow{ds_z} = \rho d\phi d\rho\, \hat{a}_z \quad \text{..............(1.26)}$$

Differential volume,

$$dv = \rho d\rho d\phi dz \quad \text{.................(1.27)}$$

Fig 1.8 : Differential Volume Element in Cylindrical Coordinates

Transformation between Cartesian and Cylindrical coordinates:

Let us consider $\vec{A} = \hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z$ is to be expressed in Cartesian co-ordinate as $\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$. In doing so we note that $A_x = \vec{A}.\hat{a}_x = \left(\hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z\right).\hat{a}_x$ and it applies for other components as well.

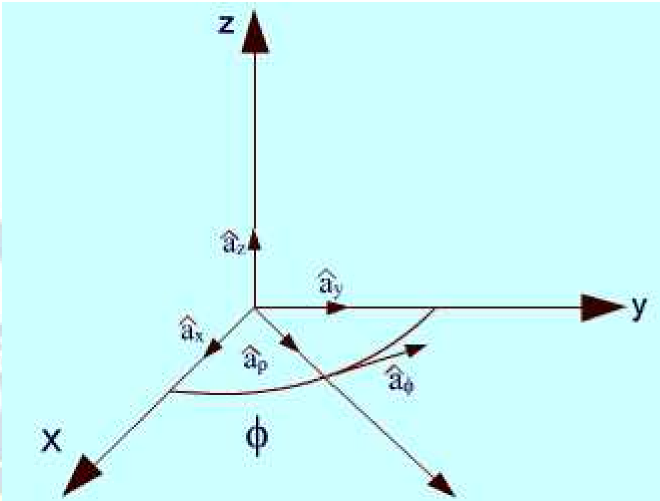


Fig 1.9 : Unit Vectors in Cartesian and Cylindrical Coordinates

$$\hat{a}_\rho.\hat{a}_x = \cos\phi$$

$$\hat{a}_\rho.\hat{a}_y = \sin\phi$$

$$\hat{a}_\phi.\hat{a}_x = \cos(\phi + \frac{\pi}{2}) = -\sin\phi$$

$$\hat{a}_\phi.\hat{a}_y = \cos\phi \qquad ....(1.28)$$

Therefore we can write,

$$A_x = \vec{A}.\hat{a}_x = A_\rho \cos\phi - A_\phi \sin\phi$$

$$A_y = \vec{A}\,\hat{a}_y = A_\rho \sin\phi + A_\phi \cos\phi$$

$$A_z = \vec{A}\,\hat{a}_z = A_z \qquad ....(1.29)$$

These relations can be put conveniently in the matrix form as:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \quad ....(1.30)$$

A_ρ, A_ϕ and A_z themselves may be functions of ρ, ϕ and z as:

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z \quad ....(1.31)$$

The inverse relationships are:

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\frac{y}{x}$$
$$z = z \quad ....(1.32)$$

SPHERICAL POLAR COORDINATE SYSTEM:

Thus we see that a vector in one coordinate system is transformed to another coordinate system through two-step process: Finding the component vectors and then variable transformation.

Spherical Polar Coordinates:

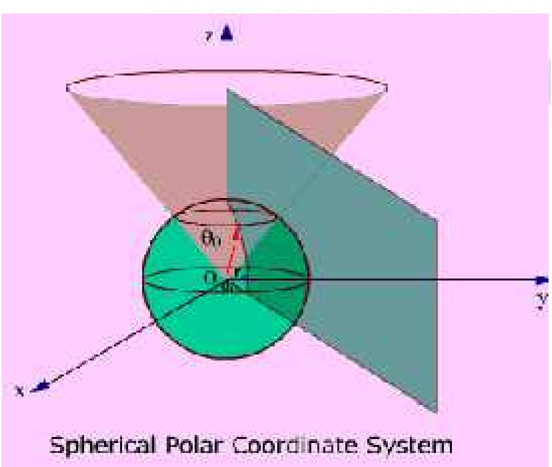


Fig 1.10: Spherical Polar Coordinate System

For spherical polar coordinate system, we have, $(u, v, w) = (r, \theta, \phi)$. A point $P(r_0, \theta_0, \phi_0)$ is represented as the intersection of

(i) Spherical surface $r=r_0$

(ii) Conical surface $\theta = \theta_0$,and

(iii) half plane containing z-axis making angle $\phi = \phi_0$ with the *xz* plane as shown in the figure 1.10.

LECTURE NOTES

The unit vectors satisfy the following relationships:

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$$
$$\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r$$
$$\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta \quad \text{.....................................(1.33)}$$

The orientation of the unit vectors are shown in the figure 1.11.

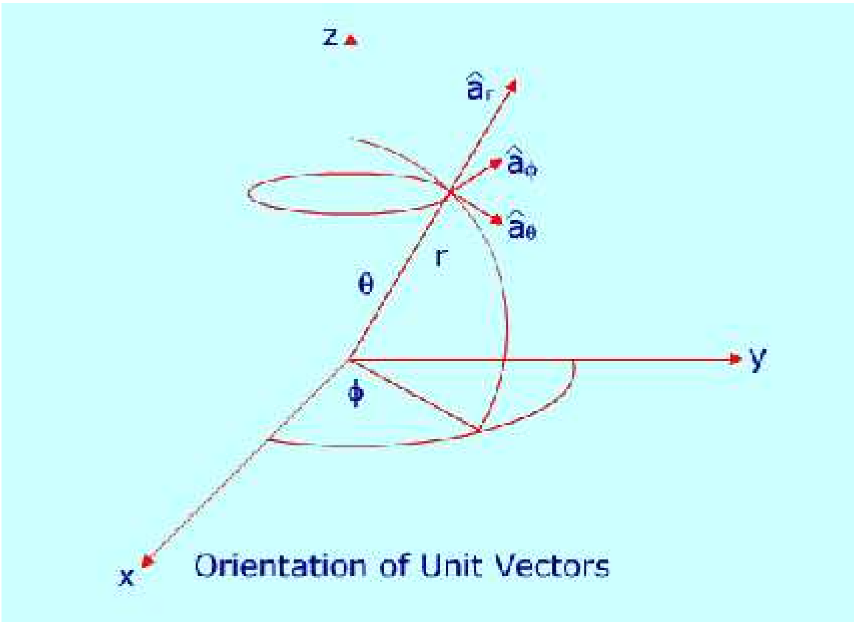


Fig 1.11: Orientation of Unit Vectors

A vector in spherical polar co-ordinates is written as : $\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$

and $d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_\phi r \sin\theta d\phi$

For spherical polar coordinate system we have h_1=1, h_2= r and h_3=$r \sin\theta$.

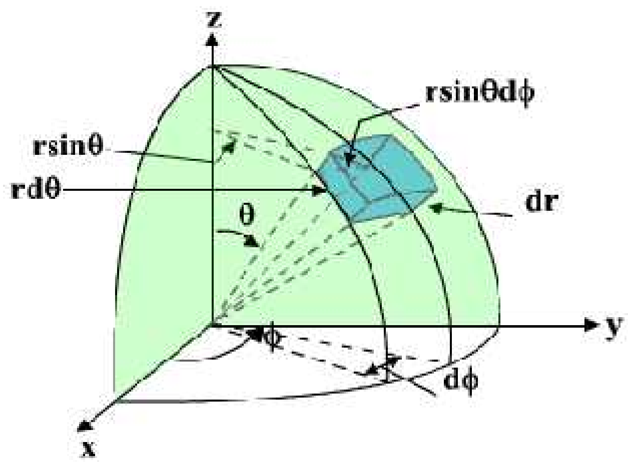


Fig 1.12(a) : Differential volume in s-p coordinates

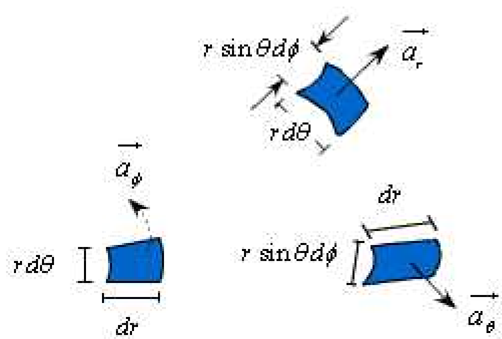


Fig 1.12(b) : Exploded view

With reference to the Figure 1.12, the elemental areas are:

$$ds_r = r^2 \sin\theta d\theta d\phi \hat{a}_r$$

$$ds_\theta = r\sin\theta dr d\phi \hat{a}_\theta$$

$$ds_\phi = r dr d\theta \hat{a}_\phi \quad \text{.......................(1.34)}$$

and elementary volume is given by

$$d\upsilon = r^2 \sin\theta dr d\theta d\phi \quad \text{........................(1.35)}$$

Coordinate transformation between rectangular and spherical polar:

With reference to the figure 1.13 ,we can write the following equations:

$$\hat{a}_r \cdot \hat{a}_x = \sin\theta\cos\phi$$

$$\hat{a}_r \cdot \hat{a}_y = \sin\theta\sin\phi$$

$$\hat{a}_r \cdot \hat{a}_z = \cos\theta$$

$$\hat{a}_\theta \cdot \hat{a}_x = \cos\theta\cos\phi$$

$$\hat{a}_\theta \cdot \hat{a}_y = \cos\theta\sin\phi$$

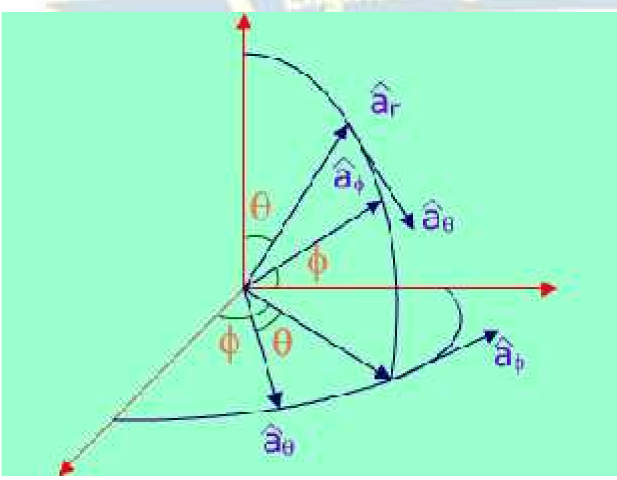
$$\hat{a}_\theta \cdot \hat{a}_z = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$$

$$\hat{a}_\phi \cdot \hat{a}_x = \cos(\phi + \frac{\pi}{2}) = -\sin\phi$$

$$\hat{a}_\phi \cdot \hat{a}_y = \cos\phi$$

$$\hat{a}_\phi \cdot \hat{a}_z = 0$$

......................................(1.36)



Given a vector $\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$ in the spherical polar coordinate system, its component in the cartesian coordinate system can be found out as follows:

Fig 1.13: Coordinate transformation

$$A_x = \vec{A}.\hat{a}_x = A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$$ ....(1.37)

Similarly,

$$A_y = \vec{A}\,\hat{a}_y = A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi$$ ....(1.38a)

$$A_z = \vec{A}\,\hat{a}_z = A_r \cos\theta - A_\theta \sin\theta$$ ....(1.38b)

The above equation can be put in a compact form:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$ ....(1.39)

The components $A_r, A_\theta \text{ and } A_\phi$ themselves will be functions of $r, \theta \text{ and } \phi$. $r, \theta \text{ and } \phi$ are related to x,y and z as:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$ ....(1.40)

and conversely,

$$r = \sqrt{x^2 + y^2 + z^2}$$ ....(1.41a)

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}$$ ....(1.41b)

$$\phi = \tan^{-1}\frac{y}{x}$$ ....(1.41c)

Using the variable transformation listed above, the vector components, which are functions of variables of one coordinate system, can be transformed to functions of variables of other coordinate system and a total transformation can be done.

Line, surface and volume integrals

In electromagnetic theory, we come across integrals, which contain vector functions. Some representative integrals are listed below:

$$\int_V \vec{F}dv \qquad \int_C \phi d\vec{l} \qquad \int_C \vec{F}.d\vec{l} \qquad \int_S \vec{F}.d\vec{s}$$

In the above integrals, $\vec{F}$ and ϕ respectively represent vector and scalar function of space coordinates. *C*,*S* and *V* represent path, surface and volume of integration. All these integrals are evaluated using extension of the usual one-dimensional integral as the limit of a sum, i.e., if a function *f*(*x*) is defined over arrange *a* to *b* of values of *x*, then the integral is given by

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f_i \delta x_i \quad ....(1.42)$$

where the interval (*a*,*b*) is subdivided into n continuous interval of lengths $\delta x_1, \ldots\ldots\ldots, \delta x_n$.

Line Integral: Line integral $\int_C \vec{E}\cdot\vec{dl}$ is the dot product of a vector with a specified *C*; in other words it is the integral of the tangential component $\vec{E}$ along the curve *C*.

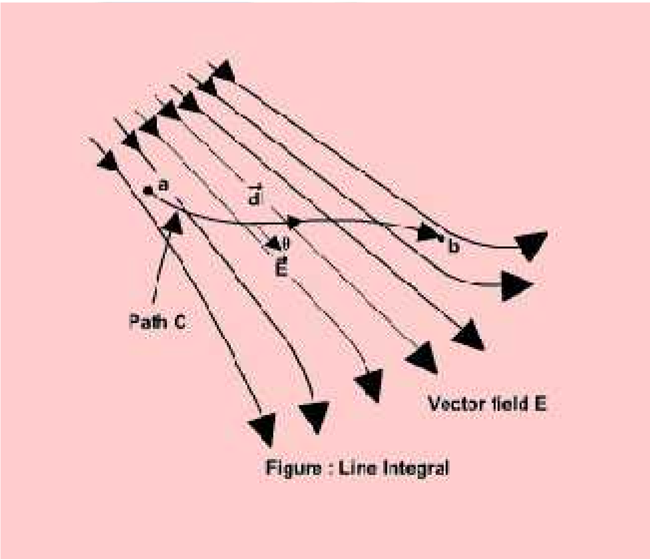


Fig 1.14: Line Integral

As shown in the figure 1.14, given a vector $\vec{E}$ around *C*, we define the integral $\int_C \vec{E}.d\vec{l} = \int_a^b E\cos\theta dl$ as the line integral of E along the curve C.

If the path of integration is a closed path as shown in the figure the line integral becomes a closed line integral and is called the circulation of $\vec{E}$ around C and denoted as $\oint_C \vec{E}.d\vec{l}$ as shown in the figure 1.15.

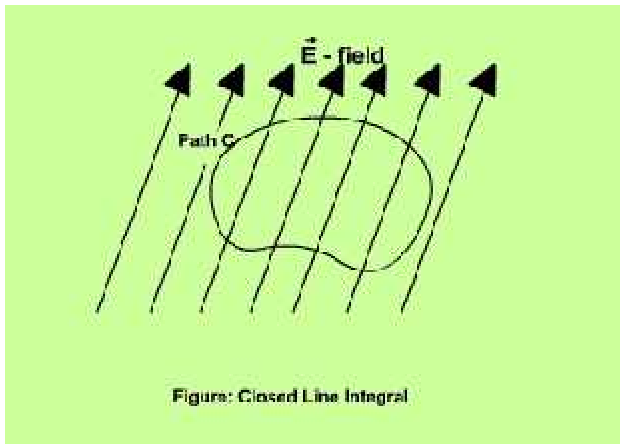


Fig 1.15: Closed Line Integral

Surface Integral :

Given a vector field $\vec{A}$, continuous in a region containing the smooth surface S, we define the surface integral or the flux of $\vec{A}$ through S as $\psi = \int_S A\cos\theta dS = \int_S \vec{A}.\hat{a}_n\, dS = \int_S \vec{A}\, d\vec{S}$ as surface integral over surface S.

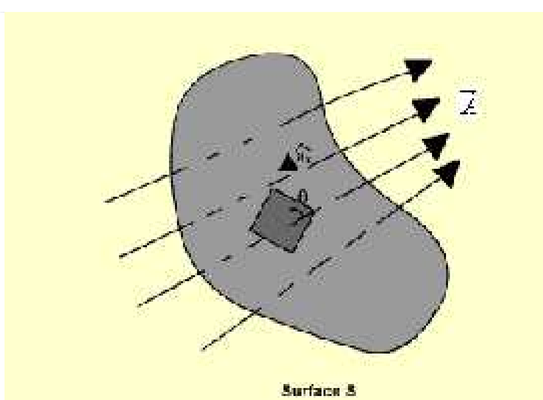


Fig 1.16 : Surface Integral

If the surface integral is carried out over a closed surface, then we write $\psi = \oint_S \vec{A}\, d\vec{S}$

Volume Integrals:

We define $\int_V f dV$ or $\iiint_V f dV$ as the volume integral of the scalar function *f*(function of spatial coordinates) over the volume *V*. Evaluation of integral of the form $\int_V \vec{F} dV$ can be carried out as a sum of three scalar volume integrals, where each scalar volume integral is a component of the vector $\vec{F}$

The Del Operator :

The vector differential operator ∇ was introduced by Sir W. R. Hamilton and later on developed by P. G. Tait.

Mathematically the vector differential operator can be written in the general form as:

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial}{\partial v}\hat{a}_v + \frac{1}{h_3}\frac{\partial}{\partial w}\hat{a}_w \quad \text{.................................(1.43)}$$

In Cartesian coordinates:

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \quad \text{..(1.44)}$$

In cylindrical coordinates:

$$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z \quad \text{...(1.45)}$$

and in spherical polar coordinates:

$$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi \quad \text{.................................(1.46)}$$

Gradient of a Scalar function:

Let us consider a scalar field *V*(*u*,*v*,*w*) , a function of space coordinates.

Gradient of the scalar field *V* is a vector that represents both the magnitude and direction of the maximum space rate of increase of this scalar field *V*.

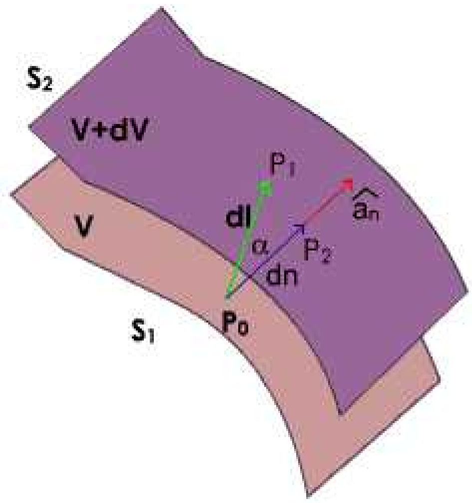


Fig 1.17 : Gradient of a scalar function

As shown in figure 1.17, let us consider two surfaces $S_1$and S_2 where the function V has constant magnitude and the magnitude differs by a small amount dV. Now as one moves from S_1 to S_2, the magnitude of spatial rate of change of V i.e. dV/dl depends on the direction of elementary path length dl, the maximum occurs when one traverses from $S_1$to S_2along a path normal to the surfaces as in this case the distance is minimum.

By our definition of gradient we can write:

$$\text{grad}V = \frac{dV}{dn}\hat{a}_n = \nabla V \quad \text{...(1.47)}$$

since $d\vec{n}$ which represents the distance along the normal is the shortest distance between the two surfaces.

For a general curvilinear coordinate system

$$d\vec{l} = \hat{a}_u\, dl_u + \hat{a}_v\, dl_v + \hat{a}_w\, dl_w = \left(h_1 du\, \hat{a}_u + h_2 dv\, \hat{a}_v + h_3 dw\, \hat{a}_w\right) \quad \text{....................(1.48)}$$

Further we can write

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha = \nabla V \cdot \hat{a}_l \quad \text{..(1.49)}$$

Hence,

$$dV = \nabla V.dl = \nabla V.(h_1 du\,\hat{a}_u + h_2 dv\,\hat{a}_v + h_3 dw\,\hat{a}_w)$$ ....(1.50)

Also we can write,

$$dV = \frac{\partial V}{\partial l_u} dl_u + \frac{\partial V}{\partial l_v} dl_v + \frac{\partial V}{\partial l_w} dl_w$$
$$= \left(\frac{\partial V}{\partial l_u}\hat{a}_u + \frac{\partial V}{\partial l_v}\hat{a}_v + \frac{\partial V}{\partial l_w}\hat{a}_w\right)\cdot(dl_u\hat{a}_u + dl_v\hat{a}_v + dl_w\hat{a}_w)$$
$$= \left(\frac{\partial V}{h_1\partial u}\hat{a}_u + \frac{\partial V}{h_2\partial v}\hat{a}_v + \frac{\partial V}{h_3\partial w}\hat{a}_w\right)\cdot(h_1 du\hat{a}_u + h_2 dv\hat{a}_v + h_3 dw\hat{a}_w)$$ ....(1.51)

By comparison we can write,

$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial V}{\partial v}\hat{a}_v + \frac{1}{h_3}\frac{\partial V}{\partial w}\hat{a}_w$$ ....(1.52)

Hence for the Cartesian, cylindrical and spherical polar coordinate system, the expressions for gradient can be written as:
In Cartesian coordinates:

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$ ....(1.53)

In cylindrical coordinates:

$$\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z$$ ....(1.54)

and in spherical polar coordinates:

$$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$$ ....(1.55)

The following relationships hold for gradient operator.

$$\nabla(U+V) = \nabla U + \nabla V$$
$$\nabla(UV) = V\nabla U + U\nabla V$$
$$\nabla\left(\frac{U}{V}\right) = \frac{V\nabla U - U\nabla V}{V^2}$$
$$\nabla V^n = nV^{n-1}\nabla V \quad \text{.........(1.56)}$$

where U and V are scalar functions and n is an integer.

It may further be noted that since magnitude of $\frac{dV}{dl}(=\Delta V.\hat{a}_1)$ depends on the direction of dl, it is called the **directional derivative**. If $A = \Delta V,\ V$ is called the scalar potential function of the vector function $\vec{A}$.

Divergence of a Vector Field:

In study of vector fields, directed line segments, also called flux lines or streamlines, represent field variations graphically. The intensity of the field is proportional to the density of lines. For example, the number of flux lines passing through a unit surface S normal to the vector measures the vector field strength.

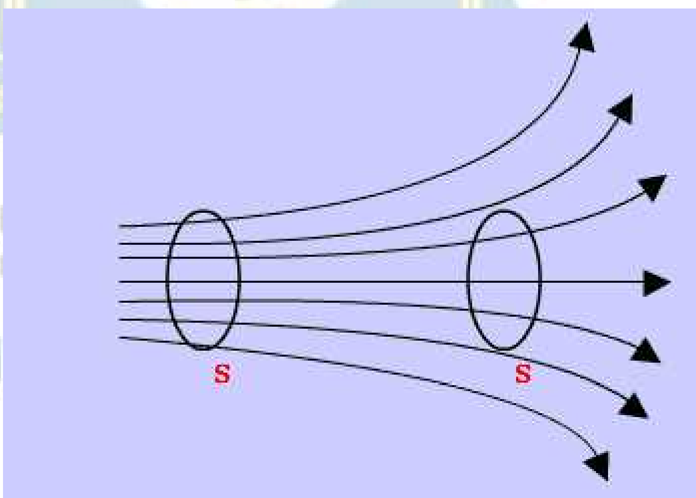


Fig 1.18: Flux Lines

We have already defined flux of a vector field as

$$\psi = \int_S A\cos\theta ds = \int_S \vec{A}\cdot\hat{a}_n ds = \int_S \vec{A}\cdot d\vec{s} \quad \text{.........(1.57)}$$

For a volume enclosed by a surface,

$$\psi = \oint_S \vec{A} \cdot d\vec{s} \quad \text{.........(1.58)}$$

We define the divergence of a vector field $\vec{A}$ at a point *P* as the net outward flux from a volume enclosing *P*, as the volume shrinks to zero.

$$div\ \vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A} \cdot d\vec{s}}{\Delta v} \quad \text{.........(1.59)}$$

Here ΔV is the volume that encloses *P* and *S* is the corresponding closed surface.

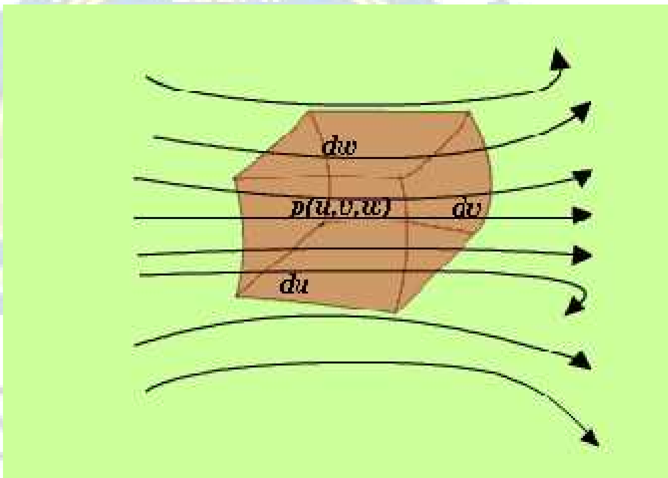


Fig 1.19: Evaluation of divergence in curvilinear coordinate

Let us consider a differential volume centered on point P(*u*,*v*,*w*) in a vector field $\vec{A}$.

The flux through an elementary area normal to *u* is given by ,

$$\phi_u = \vec{A} \cdot \hat{a}_u h_2 h_3 dv dw \quad \text{.........(1.60)}$$

Net outward flux along *u* can be calculated considering the two elementary surfaces perpendicular to *u* .

$$\left[h_2h_3A_u\Big|_{\left(u+\frac{du}{2},v,w\right)} - h_2h_3A_u\Big|_{\left(u-\frac{du}{2},v,w\right)} \right] dvdw \cong \frac{\partial\left(h_2h_3A_u\right)}{\partial u}dudvdw$$

....(1.61)

Considering the contribution from all six surfaces that enclose the volume, we can write

$$\text{div}\,\vec{A} = \nabla\cdot\vec{A} = \lim_{\Delta v\to 0}\frac{\oint_s \vec{A}\cdot\vec{ds}}{\Delta v} = \frac{dudvdw\frac{\partial\left(h_2h_3A_u\right)}{\partial u} + dudvdw\frac{\partial\left(h_1h_3A_v\right)}{\partial v} + dudvdw\frac{\partial\left(h_1h_2A_w\right)}{\partial w}}{h_1h_2h_3dudvdw}$$

$$\therefore\ \nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial\left(h_2h_3A_u\right)}{\partial u} + \frac{\partial\left(h_1h_3A_v\right)}{\partial v} + \frac{\partial\left(h_1h_2A_w\right)}{\partial w}\right]$$

....(1.62)

Hence for the Cartesian, cylindrical and spherical polar coordinate system, the expressions for divergence can be written as:

In Cartesian coordinates:

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ ....(1.63)

In cylindrical coordinates:

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial\left(\rho A_\rho\right)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$ ....(1.64)

and in spherical polar coordinates:

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$ ....(1.65)

In connection with the divergence of a vector field, the following can be noted

- Divergence of a vector field gives a scalar.

$$\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$$

$$\nabla\cdot(V\vec{A}) = V\nabla\cdot\vec{A} + \vec{A}\cdot\nabla V$$ ....(1.66)

Divergence theorem :

Divergence theorem states that the volume integral of the divergence of vector field is equal to the net outward flux of the vector through the closed surface that bounds the volume. Mathematically, $\int_V \nabla \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{s}$

Proof:

Let us consider a volume V enclosed by a surface S . Let us subdivide the volume in large number of cells. Let the k^{th} cell has a volume ΔV_K and the corresponding surface is denoted by S_k. Interior to the volume, cells have common surfaces. Outward flux through these common surfaces from one cell becomes the inward flux for the neighboring cells. Therefore when the total flux from these cells are considered, we actually get the net outward flux through the surface surrounding the volume. Hence we can write:

$$\oint_S \vec{A} \cdot d\vec{s} = \sum_k \oint_{S_k} \vec{A} \cdot d\vec{s} = \sum_k \frac{\oint_{S_k} \vec{A} \cdot d\vec{s}}{\Delta V_k} \Delta V_k \quad \text{......................................(1.67)}$$

In the limit, that is when $K \to \infty$ and $\Delta V_K \to 0$ the right hand of the expression can be written as $\int_V \nabla . A dV$.

Hence we get $\oint_S \vec{A} \cdot d\vec{S} = \int_V \nabla \cdot A dV$, which is the divergence theorem.

Curl of a vector field:

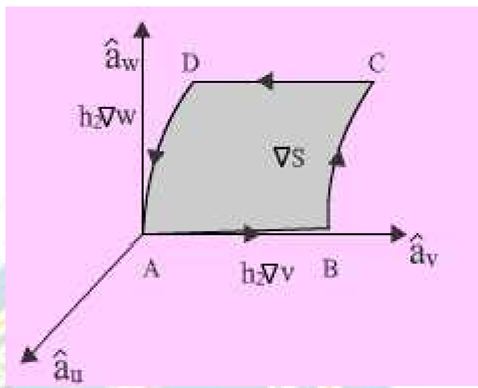
We have defined the circulation of a vector field A around a closed path as $\oint_l \vec{A} \cdot d\vec{l}$.

Curl of a vector field is a measure of the vector field's tendency to rotate about a point. Curl $\vec{A}$, also written as $\nabla \times \vec{A}$ is defined as a vector whose magnitude is maximum of the net circulation per unit area when the area tends to zero and its direction is the normal direction to the area when the area is oriented in such a way so as to make the circulation maximum.

Therefore, we can write:

$$Curl\ \vec{A} = \nabla \times \vec{A} = \lim_{\Delta S \to 0} \frac{\hat{a}_n}{\Delta S} \left[\oint_l \vec{A} \cdot dl \right]_{max} \quad \text{......................................(1.68)}$$

To derive the expression for curl in generalized curvilinear coordinate system, we first compute $\nabla \times \vec{A}.\hat{a}_u$ and to do so let us consider the figure 1.20 :



If C_1 represents the boundary of ΔS, then we can write

$$\oint_{C_1} \vec{A}\cdot d\vec{l} = \int_{AB} \vec{A}\cdot d\vec{l} + \int_{BC} \vec{A}\cdot d\vec{l} + \int_{CD} \vec{A}\cdot d\vec{l} + \int_{DA} \vec{A}\cdot d\vec{l} \quad ....(1.69)$$

The integrals on the RHS can be evaluated as follows:

$$\int_{AB} \vec{A}\cdot d\vec{l} = (A_u\hat{a}_u + A_v\hat{a}_v + A_w\hat{a}_w)\cdot h_2\Delta v\hat{a}_v = A_v h_2 \Delta v \quad ....(1.70)$$

$$\int_{CD} \vec{A}\cdot d\vec{l} = -\left(A_v h_2 \Delta v + \frac{\partial}{\partial w}(A_v h_2 \Delta v)\Delta w\right) \quad ....(1.71)$$

The negative sign is because of the fact that the direction of traversal reverses. Similarly,

$$\int_{BC} \vec{A}\cdot d\vec{l} = \left(A_w h_3 \Delta w + \frac{\partial}{\partial v}(A_w h_3 \Delta w)\Delta v\right) \quad ....(1.72)$$

$$\int_{DA} \vec{A}\cdot d\vec{l} = -A_w h_3 \Delta w \quad ....(1.73)$$

Adding the contribution from all components, we can write:

$$\oint_{C_1} \vec{A}\cdot d\vec{l} = \left(\frac{\partial}{\partial v}(A_w h_3) - \frac{\partial}{\partial w}(A_v h_3)\right)\Delta v \Delta w$$ ....(1.74)

Therefore, $$\left(\nabla\times\vec{A}\right).\hat{a}_u = \frac{\oint_{C_1} \vec{A}\cdot d\vec{l}}{h_2 h_3 \Delta v \Delta w} = \frac{1}{h_2 h_3}\left(\frac{\partial(h_3 A_w)}{\partial v} - \frac{\partial(h_2 A_v)}{\partial w}\right)$$ ....(1.75)

In the same manner if we compute for $\left(\nabla\times\vec{A}\right).\hat{a}_v$ and $\left(\nabla\times\vec{A}\right).\hat{a}_w$ we can write,

$$\nabla\times\vec{A} = \frac{1}{h_2 h_3}\left(\frac{\partial(h_3 A_w)}{\partial v} - \frac{\partial(h_2 A_v)}{\partial w}\right)\hat{a}_u + \frac{1}{h_1 h_3}\left(\frac{\partial(h_1 A_u)}{\partial w} - \frac{\partial(h_3 A_w)}{\partial u}\right)\hat{a}_v + \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 A_v)}{\partial u} - \frac{\partial(h_1 A_u)}{\partial v}\right)\hat{a}_w$$ ....(1.76)

This can be written as,

$$\nabla\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{a}_u & h_2\hat{a}_v & h_3\hat{a}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 A_u & h_2 A_v & h_3 A_w \end{vmatrix}$$ ....(1.77)

In Cartesian coordinates: $$\nabla\times\vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$ ....(1.78)

In Cylindrical coordinates, $$\nabla\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$ ....(1.79)

In Spherical polar coordinates,

$$\nabla\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$ ....(1.80)

Curl operation exhibits the following properties:

(i) Curl of a vector field is another vector field.

(ii) $\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A}+\nabla\times\vec{B}$

(iii) $\nabla\times(V\vec{A}) = \nabla V\times\vec{A}+V\nabla\times\vec{A}$

(iv) $\nabla.(\nabla\times\vec{A}) = 0$

(v) $\nabla\times\nabla V = 0$

(vi) $\nabla\times(\vec{A}\times\vec{B}) = \vec{A}\nabla.\vec{B}-\vec{B}\nabla.\vec{A}+(\vec{B}.\nabla)\vec{A}-(\vec{A}.\nabla)\vec{B}$ ....(1.81)

Stoke's theorem :

It states that the circulation of a vector field $\vec{A}$ around a closed path is equal to the integral of $\nabla\times\vec{A}$ over the surface bounded by this path. It may be noted that this equality holds provided $\vec{A}$ and $\nabla\times\vec{A}$ are continuous on the surface.

$$\oint_L \vec{A}\cdot d\vec{l} = \int_S \nabla\times\vec{A}\cdot d\vec{s}$$ ....(1.82)

Proof: Let us consider an area S that is subdivided into large number of cells as shown in the figure 1.21.

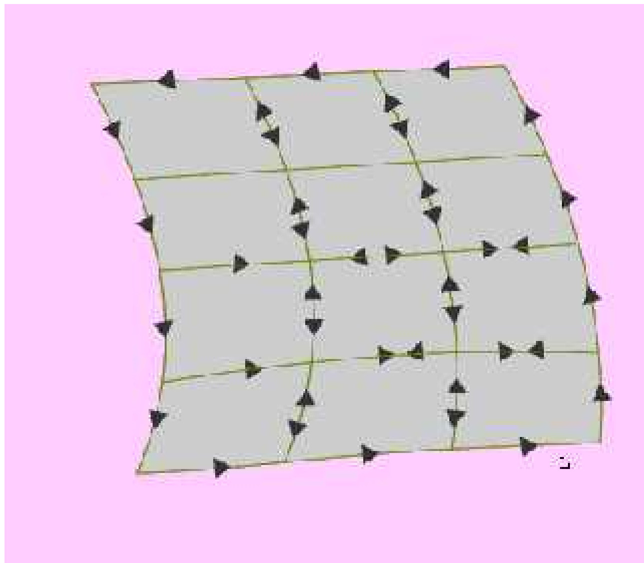


Fig 1.21: Stokes theorem

Let k^{th}cell has surface area ΔS_k and is bounded path L_k while the total area is bounded by path L. As seen from the figure that if we evaluate the sum of the line integrals around the elementary areas, there is cancellation along every interior path and we are left the line integral along path L. Therefore we can write,

$$\oint_L \vec{A}\cdot d\vec{l} = \sum_k \oint_{L_k} \vec{A}\cdot d\vec{l} = \sum_k \frac{\oint_{L_k} \vec{A}\cdot d\vec{l}}{\Delta S_k}\Delta S_k \qquad \text{.............(1.83)}$$

As $\Delta S_k \rightarrow 0$

$$\oint_L \vec{A}\cdot d\vec{l} = \int_S \nabla\times\vec{A}\cdot d\vec{s} \qquad \text{............(1.84)}$$

which is the stoke's theorem.