

Course/Branch : B.E./ECE	Year / Semester :II/IV	Format No.	NAC/TLP-07a.5
Subject Code :EC8451	Subject Name :ELECTROMAGNETIC FIELDS	Rev. No.	02
Unit No :3	Unit Name : MAGNETOSTATICS	Date	14-11-2017

**LECTURE NOTES**

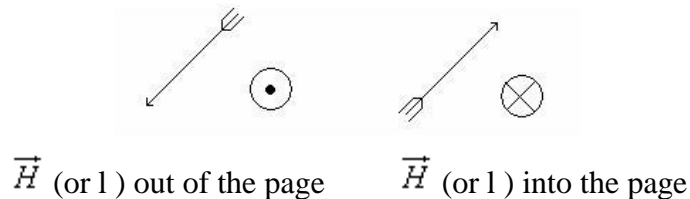
**INTRODUCTION:**

In previous chapters we have seen that an electrostatic field is produced by static or stationary charges. The relationship of the steady magnetic field to its sources is much more complicated.

The source of steady magnetic field may be a permanent magnet, a direct current or an electric field changing with time. In this chapter we shall mainly consider the magnetic field produced by a direct current. The magnetic field produced due to time varying electric field will be discussed later. Historically, the link between the electric and magnetic field was established Oersted in 1820. Ampere and others extended the investigation of magnetic effect of electricity. There are two major laws governing the magnetostatic fields are:

- Biot-Savart Law
- Ampere's Law

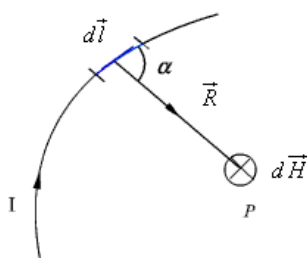
Usually, the magnetic field intensity is represented by the vector  $\vec{H}$ . It is customary to represent the direction of the magnetic field intensity (or current) by a small circle with a dot or cross sign depending on whether the field (or current) is out of or into the page as shown in Fig. 4.1.



**Fig. 4.1: Representation of magnetic field (or current)**

**BIOT-SAVART LAW**

This law relates the magnetic field intensity  $dH$  produced at a point due to a differential current element  $Id\vec{l}$  as shown in Fig. 4.2.



**Fig. 4.2: Magnetic field intensity due to a current element**

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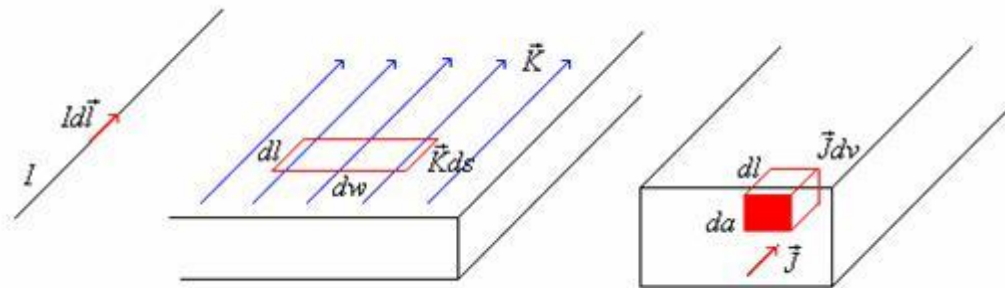
The magnetic field intensity  $d\vec{H}$  at P can be written as,

$$d\vec{H} = \frac{Id\vec{l} \times \hat{a}_R}{4\pi R^2} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3} \dots\dots\dots(4.1a)$$

$$dH = \frac{Idl \sin\alpha}{4\pi R^2} \dots\dots\dots(4.1b)$$

where  $R = |\vec{R}|$  is the distance of the current element from the point P.

Similar to different charge distributions, we can have different current distribution such as line current, surface current and volume current. These different types of current densities are shown in Fig. 4.3.



Line Current                      Surface Current                      Volume Current

**Fig. 4.3: Different types of current distributions**

By denoting the surface current density as K (in amp/m) and volume current density as J (in amp/m<sup>2</sup>) we can write:

$$Id\vec{l} = \vec{K}ds = \vec{J}dv \dots\dots\dots(4.2)$$

(It may be noted that  $I = Kdw = Jda$  )

Employing Biot-Savart Law, we can now express the magnetic field intensity H. In terms of these current distributions.

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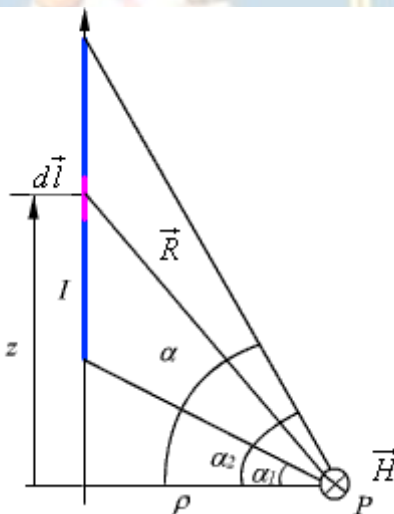
$$\vec{H} = \int \frac{I d\vec{l} \times \vec{R}}{4\pi R^3} \dots\dots\dots \text{for line current} \dots\dots\dots (4.3a)$$

$$\vec{H} = \int \frac{K ds \times \vec{R}}{4\pi R^3} \dots\dots\dots \text{for surface current} \dots\dots\dots (4.3b)$$

$$\vec{H} = \int \frac{\vec{J} dv \times \vec{R}}{4\pi R^3} \dots\dots\dots \text{for volume current} \dots\dots\dots (4.3c)$$

To illustrate the application of Biot - Savart's Law, we consider the following example.

**Example 4.1:** We consider a finite length of a conductor carrying a current  $I$  placed along z-axis as shown in the Fig 4.4. We determine the magnetic field at point P due to this current carrying conductor.



**Fig. 4.4: Field at a point P due to a finite length current carrying conductor**

With reference to Fig. 4.4, we find that

$$d\vec{l} = dz \hat{a}_z \text{ and } \vec{R} = \rho \hat{a}_\rho - z \hat{a}_z \dots\dots\dots (4.4)$$

Applying Biot - Savart's law for the current element  $I d\vec{l}$

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we can write,

$$\vec{dH} = \frac{I d\vec{l} \times \vec{R}}{4\pi R^3} = \frac{\rho dz \hat{a}_\phi}{4\pi[\rho^2 + z^2]^{3/2}} \dots\dots\dots(4.5)$$

Substituting  $\frac{z}{\rho} = \tan \alpha$  we can write,

$$\vec{H} = \int_{\alpha_1}^{\alpha_2} \frac{I \rho^2 \sec^2 \alpha d\alpha}{4\pi \rho^3 \sec^3 \alpha} \hat{a}_\phi = \frac{I}{4\pi \rho} (\sin \alpha_2 - \sin \alpha_1) \hat{a}_\phi \dots\dots\dots(4.6)$$

We find that, for an infinitely long conductor carrying a current I,  $\alpha_2 = 90^\circ$  and  $\alpha_1 = -90^\circ$

Therefore,  $\vec{H} = \frac{I}{2\pi \rho} \hat{a}_\phi \dots\dots\dots(4.7)$

**AMPERE'S CIRCITAL LAW :**

Ampere's circuital law states that the line integral of the magnetic field  $\vec{H}$  (circulation of H) around a closed path is the net current enclosed by this path. Mathematically,

$$\oint \vec{H} \cdot d\vec{l} = I_{enc} \dots\dots\dots(4.8)$$

The total current  $I_{enc}$  can be written as,

$$I_{enc} = \int_V \vec{J} \cdot d\vec{s} \dots\dots\dots(4.9)$$

By applying Stoke's theorem, we can write

$$\begin{aligned} \oint \vec{H} \cdot d\vec{l} &= \int_V \nabla \times \vec{H} \cdot d\vec{s} \\ \therefore \int_V \nabla \times \vec{H} \cdot d\vec{s} &= \int_V \vec{J} \cdot d\vec{s} \\ \therefore \nabla \times \vec{H} &= \vec{J} \dots\dots\dots(4.10) \end{aligned}$$

which is the Ampere's law in the point form.

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**APPLICATIONS OF AMPERE'S LAW:**

We illustrate the application of Ampere's Law with some examples.

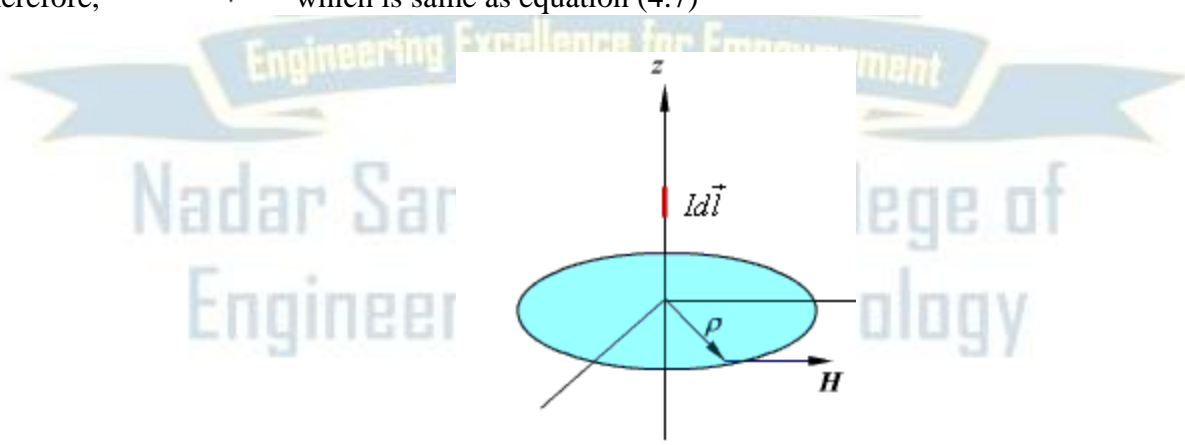
**Example 4.2:** We compute magnetic field due to an infinitely long thin current carrying conductor as shown in Fig. 4.5. Using Ampere's Law, we consider the close path to be a circle of radius  $\rho$  as shown in the Fig. 4.5.

If we consider a small current element  $Id\vec{l}(= Idz\hat{a}_z)$ ,  $d\vec{H}$  is perpendicular to the plane containing both  $d\vec{l}$  and  $\vec{R}(= \rho\hat{a}_\rho)$ . Therefore only component of  $\vec{H}$  that will be present is  $H_\phi$ , i.e.,  $\vec{H} = H_\phi\hat{a}_\phi$ .

By applying Ampere's law we can write,

$$\vec{H} = \frac{I}{2\pi\rho} \hat{a}_\phi \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho 2\pi = I \dots\dots\dots(4.11)$$

Therefore,  $\vec{H} = \frac{I}{2\pi\rho} \hat{a}_\phi$  which is same as equation (4.7)



**Fig. 4.5: Magnetic field due to an infinite thin current carrying conductor**

**Example 4.3:** We consider the cross section of an infinitely long coaxial conductor, the inner conductor carrying a current I and outer conductor carrying current - I as shown in figure 4.6. We compute the magnetic field as a function of  $\rho$  as follows:

In the region  $0 \leq \rho \leq R_1$

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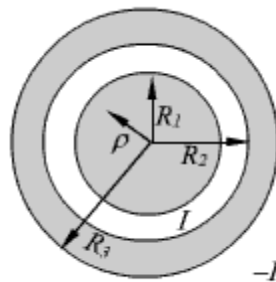
$$I_{enc} = I \frac{\rho^2}{R_1^2} \dots\dots\dots(4.12)$$

$$H_\phi = \frac{I_{enc}}{2\pi\rho} = \frac{I\rho}{2\pi a^2} \dots\dots\dots(4.13)$$

In the region  $R_1 \leq \rho \leq R_2$

$$I_{enc} = I$$

$$H_\phi = \frac{I}{2\pi\rho} \dots\dots\dots(4.14)$$



**Fig. 4.6: Coaxial conductor carrying equal and opposite currents**

In the region  $R_2 \leq \rho \leq R_3$

$$I_{enc} = I - I \frac{\rho^2 - R_2^2}{R_3^2 - R_2^2} \dots\dots\dots(4.15)$$

$$H_\phi = \frac{I}{2\pi\rho} \frac{R_3^2 - \rho^2}{R_3^2 - R_2^2} \dots\dots\dots(4.16)$$

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In the region  $\rho > R_3$

$$I_{enc} = 0 \quad H_\phi = 0 \quad \dots\dots\dots(4.17)$$

**MAGNETIC FLUX DENSITY:**

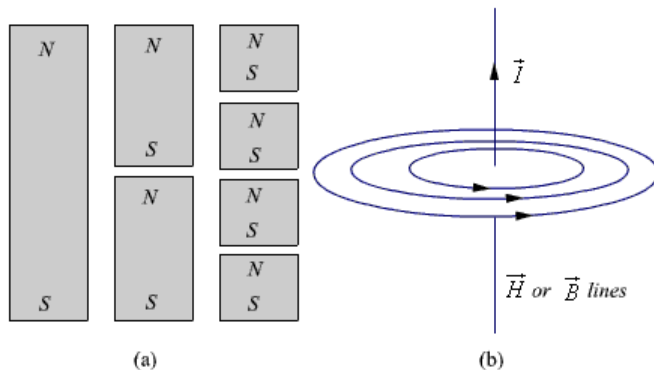
In simple matter, the magnetic flux density  $\vec{B}$  related to the magnetic field intensity  $\vec{H}$  as  $\vec{B} = \mu\vec{H}$  where  $\mu$  called the permeability. In particular when we consider the free space  $\vec{B} = \mu_0\vec{H}$  where  $\mu_0 = 4\pi \times 10^{-7}$  H/m is

the permeability of the free space. Magnetic flux density is measured in terms of Wb/m<sup>2</sup>.

The magnetic flux density through a surface is given by:

$$\psi = \int_S \vec{B} \cdot d\vec{s} \quad \text{Wb} \quad \dots\dots\dots(4.18)$$

In the case of electrostatic field, we have seen that if the surface is a closed surface, the net flux passing through the surface is equal to the charge enclosed by the surface. In case of magnetic field isolated magnetic charge (i. e. pole) does not exist. Magnetic poles always occur in pair (as N-S). For example, if we desire to have an isolated magnetic pole by dividing the magnetic bar successively into two, we end up with pieces each having north (N) and south (S) pole as shown in Fig. 4.7 (a). This process could be continued until the magnets are of atomic dimensions; still we will have N-S pair occurring together. This means that the magnetic poles cannot be isolated.



**Fig. 4.7: (a) Subdivision of a magnet (b) Magnetic field/ flux lines of a straight current carrying conductor**

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Similarly if we consider the field/flux lines of a current carrying conductor as shown in Fig. 4.7 (b), we find that these lines are closed lines, that is, if we consider a closed surface, the number of flux lines that would leave the surface would be same as the number of flux lines that would enter the surface.

From our discussions above, it is evident that for magnetic field,

$$\oint_S \vec{B} \cdot d\vec{s} = 0 \dots\dots\dots(4.19)$$

which is the Gauss's law for the magnetic field.

By applying divergence theorem, we can write:

$$\oint_S \vec{B} \cdot d\vec{s} = \int_V \nabla \cdot \vec{B} dv = 0$$

Hence,  $\nabla \cdot \vec{B} = 0 \dots\dots\dots(4.20)$

which is the Gauss's law for the magnetic field in point form.

**MAGNETIC SCALAR AND VECTOR POTENTIALS:**

In studying electric field problems, we introduced the concept of electric potential that simplified the computation of electric fields for certain types of problems. In the same manner let us relate the magnetic field intensity to a **scalar magnetic potential** and write:

$$\vec{H} = -\nabla V_m \dots\dots\dots(4.21)$$

From Ampere's law , we know that

$$\nabla \times \vec{H} = \vec{J} \dots\dots\dots(4.22)$$

Therefore,  $\nabla \times (-\nabla V_m) = \vec{J} \dots\dots\dots(4.23)$

But using vector identity,  $\nabla \times (\nabla V) = 0$  we find that  $\vec{H} = -\nabla V_m$  is valid only where  $\vec{J} = 0$ . Thus the scalar magnetic potential is defined only in the region where  $\vec{J} = 0$ . Moreover,  $V_m$  in general is not a single valued



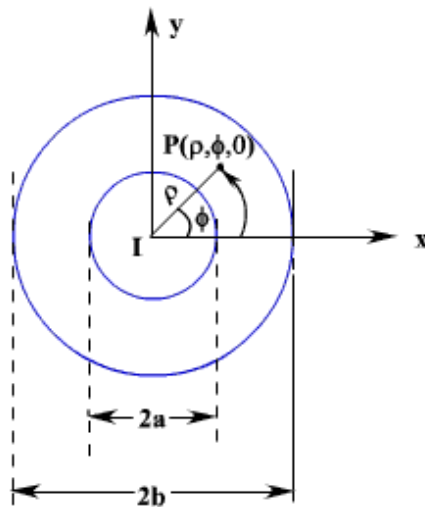
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function of position.

This point can be illustrated as follows. Let us consider the cross section of a coaxial line as shown in fig 4.8.

In the region  $a < \rho < b$ ,  $\vec{J} = 0$  and  $\vec{H} = \frac{I}{2\pi\rho} \hat{a}_\phi$



**Fig. 4.8: Cross Section of a Coaxial Line**

If  $V_m$  is the magnetic potential then,

$$-\nabla V_m = -\frac{1}{\rho} \frac{\partial V_m}{\partial \phi}$$

$$= \frac{I}{2\pi\rho}$$

$$\therefore V_m = -\frac{I}{2\pi} \phi + c$$

If we set  $V_m = 0$  at  $\phi = 0$  then  $c=0$  and  $V_m = -\frac{I}{2\pi} \phi$

$$\therefore \text{At } \phi = \phi_0 \quad V_m = -\frac{I}{2\pi} \phi_0$$

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We observe that as we make a complete lap around the current carrying conductor , we reach  $\phi_0$  again but  $V_m$  this time becomes

$$V_m = -\frac{I}{2\pi}(\phi_0 + 2\pi)$$

We observe that value of  $V_m$  keeps changing as we complete additional laps to pass through the same point. We introduced  $V_m$  analogous to electrostatic potential  $V$ . But for static electric fields,  $\nabla \times \vec{E} = 0$  and  $\oint \vec{E} \cdot d\vec{l} = 0$ , whereas for steady magnetic field  $\nabla \times \vec{H} = \vec{J}$  wherever  $\vec{J} = 0$  but  $\oint \vec{H} \cdot d\vec{l} = I$  even if  $\vec{J} = 0$  along the path of integration.

We now introduce the **vector magnetic potential** which can be used in regions where current density may be zero or nonzero and the same can be easily extended to time varying cases. The use of vector magnetic potential provides elegant ways of solving EM field problems.

Since  $\nabla \cdot \vec{B} = 0$  and we have the vector identity that for any vector  $\vec{A}$ ,  $\nabla \cdot (\nabla \times \vec{A}) = 0$ , we can write  $\vec{B} = \nabla \times \vec{A}$ .

Here, the vector field  $\vec{A}$  is called the vector magnetic potential. Its SI unit is Wb/m. Thus if can find  $\vec{A}$  of a given current distribution,  $\vec{B}$  can be found from  $\vec{A}$  through a curl operation.

We have introduced the vector function  $\vec{A}$  and related its curl to  $\vec{B}$ . A vector function is defined fully in terms of its curl as well as divergence. The choice of  $\nabla \cdot \vec{A}$  is made as follows.

$$\nabla \times \nabla \times \vec{A} = \mu \nabla \times \vec{H} = \mu \vec{J} \dots\dots\dots(4.24)$$

By using vector identity,  $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \dots\dots\dots(4.25)$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} \dots\dots\dots(4.26)$$

Great deal of simplification can be achieved if we choose  $\nabla \cdot \vec{A} = 0$ .

Putting  $\nabla \cdot \vec{A} = 0$ , we get  $\nabla^2 \vec{A} = -\mu \vec{J}$  which is vector poisson equation.

In Cartesian coordinates, the above equation can be written in terms of the components as

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$$\nabla^2 A_x = -\mu J_x \dots\dots\dots(4.27a)$$

$$\nabla^2 A_y = -\mu J_y \dots\dots\dots(4.27b)$$

$$\nabla^2 A_z = -\mu J_z \dots\dots\dots(4.27c)$$

The form of all the above equation is same as that of

$$\nabla^2 V = -\frac{\rho}{\epsilon} \dots\dots\dots(4.28)$$

for which the solution is

$$V = \frac{1}{4\pi\epsilon} \int_V \frac{\rho}{R} dv', \quad R = |\vec{r} - \vec{r}'| \dots\dots\dots(4.29)$$

In case of time varying fields we shall see that  $\nabla \cdot \vec{A} = \mu\epsilon \frac{\partial V}{\partial t}$ , which is known as Lorentz condition,  $V$  being the electric potential. Here we are dealing with static magnetic field, so  $\nabla \cdot \vec{A} = 0$ .

By comparison, we can write the solution for  $A_x$  as

$$A_x = \frac{\mu}{4\pi} \int_V \frac{J_x}{R} dv' \dots\dots\dots(4.30)$$

Computing similar solutions for other two components of the vector potential, the vector potential can be written as

$$\vec{A} = \frac{\mu}{4\pi} \int_V \frac{\vec{J}}{R} dv' \dots\dots\dots(4.31)$$

This equation enables us to find the vector potential at a given point because of a volume current density  $\vec{J}$ . Similarly for line or surface current density we can write

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$$\vec{A} = \frac{\mu}{4\pi} \int_V \frac{J \vec{dl}'}{R} \dots\dots\dots(4.32)$$

$$\vec{A} = \frac{\mu}{4\pi} \int_V \frac{\vec{K}}{R} ds' \text{ respectively. } \dots\dots\dots(4.33)$$

The magnetic flux  $\psi$  through a given area S is given by

$$\psi = \int_S \vec{B} \cdot d\vec{s} \dots\dots\dots(4.34)$$

Substituting  $\vec{B} = \nabla \times \vec{A}$

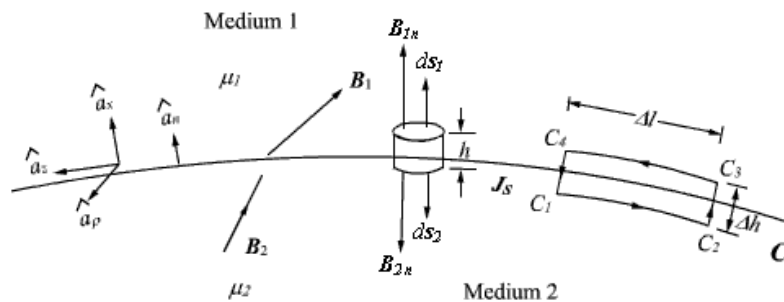
$$\psi = \int_S \nabla \times \vec{A} \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l} \dots\dots\dots(4.35)$$

Vector potential thus have the physical significance that its integral around any closed path is equal to the magnetic flux passing through that path.

**BOUNDARY CONDITION FOR MAGNETIC FIELDS:**

Similar to the boundary conditions in the electro static fields, here we will consider the behavior of  $\vec{B}$  and  $\vec{H}$  at the interface of two different media. In particular, we determine how the tangential and normal components of magnetic fields behave at the boundary of two regions having different permeabilities. The

figure 4.9 shows the interface between two media having permeabilities  $\mu_1$  and  $\mu_2$ ,  $\hat{a}_n$  being the normal vector from medium 2 to medium 1.



**Figure 4.9: Interface between two magnetic media**

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To determine the condition for the normal component of the flux density vector  $\vec{B}$ , we consider a small pill box P with vanishingly small thickness  $h$  and having an elementary area  $\Delta S$  for the faces. Over the pill box, we can write

$$\oint_S \vec{B} \cdot d\vec{s} = 0 \dots\dots\dots(4.36)$$

Since  $h \rightarrow 0$ , we can neglect the flux through the sidewall of the pill box.

$$\therefore \int_{\Delta S} \vec{B}_1 \cdot d\vec{S}_1 + \int_{\Delta S} \vec{B}_2 \cdot d\vec{S}_2 = 0 \dots\dots\dots(4.37)$$

$$d\vec{S}_1 = dS \hat{a}_n \text{ and } d\vec{S}_2 = dS \left( -\hat{a}_n \right) \dots\dots\dots(4.38)$$

$$\therefore \int_{\Delta S} B_{1n} dS - \int_{\Delta S} B_{2n} dS = 0$$

where

$$B_{1n} = \vec{B}_1 \cdot \hat{a}_n \text{ and } B_{2n} = \vec{B}_2 \cdot \hat{a}_n \dots\dots\dots(4.39)$$

Since  $\Delta S$  is small, we can write

$$(B_{1n} - B_{2n}) \Delta S = 0$$

or,  $B_{1n} = B_{2n} \dots\dots\dots(4.40)$

That is, the normal component of the magnetic flux density vector is continuous across the interface.

In vector form,

$$\hat{a}_n \cdot (\vec{B}_1 - \vec{B}_2) = 0 \dots\dots\dots(4.41)$$

To determine the condition for the tangential component for the magnetic field, we consider a closed path C as shown in figure 4.8. By applying Ampere's law we can write

$$\oint_C \vec{H} \cdot d\vec{l} = I \dots\dots\dots(4.42)$$

Since  $h \rightarrow 0$ ,

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$$\int_{c_1-c_2} \vec{H} \cdot d\vec{l} + \int_{c_3-c_4} \vec{H} \cdot d\vec{l} = I \dots\dots\dots(4.43)$$

We have shown in figure 4.8, a set of three unit vectors  $\hat{a}_n$ ,  $\hat{a}_t$  and  $\hat{a}_p$  such that they satisfy  $\hat{a}_t = \hat{a}_p \times \hat{a}_n$  (R.H. rule). Here  $\hat{a}_t$  is tangential to the interface and  $\hat{a}_p$  is the vector perpendicular to the surface enclosed by C at the interface.

The above equation can be written as

$$\vec{H}_1 \cdot \Delta \hat{a}_t - \vec{H}_2 \cdot \Delta \hat{a}_t = I = J_s \Delta l$$

or,  $H_{1t} - H_{2t} = J_s \dots\dots\dots(4.44)$

i.e., tangential component of magnetic field component is discontinuous across the interface where a free surface current exists.

If  $J_s = 0$ , the tangential magnetic field is also continuous. If one of the medium is a perfect conductor  $J_s$  exists on the surface of the perfect conductor.

In vector form we can write,

$$\begin{aligned} & (\vec{H}_1 - \vec{H}_2) \cdot \hat{a}_t \Delta l \\ &= (\vec{H}_1 - \vec{H}_2) \cdot (\hat{a}_p \times \hat{a}_n) \Delta l \\ &= J_s \Delta l = \vec{J}_s \cdot \hat{a}_p \Delta l \dots\dots\dots(4.45) \end{aligned}$$

Therefore,

$$\hat{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \dots\dots\dots(4.46)$$

**INDUCTANCE AND INDUCTOR:**

Resistance, capacitance and inductance are the three familiar parameters from circuit theory. We have

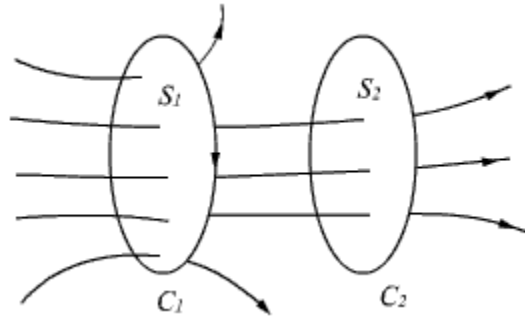
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already discussed about the parameters resistance and capacitance in the earlier chapters. In this section, we discuss about the parameter inductance. Before we start our discussion, let us first introduce the concept of flux linkage. If in a coil with N closely wound turns around where a current I produces a flux  $\phi$  and this flux links or encircles each of the N turns, the flux linkage  $\Lambda$  is defined as  $\Lambda = N\phi$ . In a linear medium, where the flux is proportional to the current, we define the self inductance L as the ratio of the total flux linkage to the current which they link.

i.e., 
$$L = \frac{\Lambda}{I} = \frac{N\phi}{I} \dots\dots\dots(4.47)$$

To further illustrate the concept of inductance, let us consider two closed loops C<sub>1</sub> and C<sub>2</sub> as shown in the figure 4.10, S<sub>1</sub> and S<sub>2</sub> are respectively the areas of C<sub>1</sub> and C<sub>2</sub>.



**Fig 4.10**

If a current I<sub>1</sub> flows in C<sub>1</sub>, the magnetic flux B<sub>1</sub> will be created part of which will be linked to C<sub>2</sub> as shown in Figure 4.10.

$$\phi_{12} = \int_{S_2} \vec{B}_1 \cdot d\vec{S}_2 \dots\dots\dots(4.48)$$

In a linear medium,  $\phi_{12}$  is proportional to I<sub>1</sub>. Therefore, we can write

$$\phi_{12} = L_{12}I_1 \dots\dots\dots(4.49)$$

where L<sub>12</sub> is the mutual inductance. For a more general case, if C<sub>2</sub> has N<sub>2</sub> turns then

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$$\Lambda_{12} = N_2\phi_{12} \dots\dots\dots(4.50)$$

and  $\Lambda_{12} = L_{12}I_1$

$$\text{or } L_{12} = \frac{\Lambda_{12}}{I_1} \dots\dots\dots(4.51)$$

i.e., the mutual inductance can be defined as the ratio of the total flux linkage of the second circuit to the current flowing in the first circuit.

As we have already stated, the magnetic flux produced in  $C_1$  gets linked to itself and if  $C_1$  has  $N_1$  turns then  $\Lambda_{11} = N_1\phi_{11}$ , where  $\phi_{11}$  is the flux linkage per turn.

Therefore, self inductance

$$L_{11}(\text{or } L \text{ as defined earlier}) = \frac{\Lambda_{11}}{I_1} \dots\dots\dots(4.52)$$

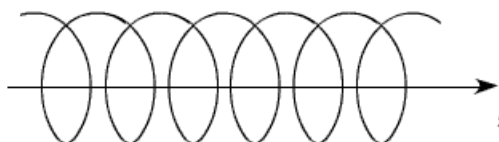
As some of the flux produced by  $I_1$  links only to  $C_1$  & not  $C_2$ .

$$\Lambda_{11} = N_1\phi_{11} > N_2\phi_{12} = \Lambda_{12} \dots\dots\dots(4.53)$$

Further in general, in a linear medium,  $L_{12} = \frac{d\Lambda_{12}}{dI_1}$  and  $L_{11} = \frac{d\Lambda_{11}}{dI_1}$

**Example 1:** Inductance per unit length of a very long solenoid:

Let us consider a solenoid having  $n$  turns/unit length and carrying a current  $I$ . The solenoid is air cored.



**Fig 4.11: A long current carrying solenoid**



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The magnetic flux density inside such a long solenoid can be calculated as

$$\vec{B} = \mu_0 n I \hat{a}_z \dots\dots\dots(4.54)$$

where the magnetic field is along the axis of the solenoid.

If S is the area of cross section of the solenoid then

$$\phi = BS = \mu_0 n I S \dots\dots\dots(4.55)$$

The flux linkage per unit length of the solenoid

$$\Lambda = n\phi = \mu_0 n^2 I S \dots\dots\dots(4.56)$$

∴ The inductance per unit length of the solenoid

$$L = \frac{\Lambda}{I} = \mu_0 n^2 S \dots\dots\dots(4.57)$$

**Example 2:** Self inductance per unit length of a coaxial cable of inner radius 'a' and outer radius 'b'. Assume a current I flows through the inner conductor.

**Solution:**

Let us assume that the current is uniformly distributed in the inner conductor so that inside the inner conductor.

i.e.,  $0 \leq \rho \leq a$

$$\begin{aligned} \vec{B}_i &= \hat{a}_\phi \frac{\mu_0 I}{2\pi\rho} \frac{\pi\rho^2}{\pi a^2} \\ &= \frac{I\mu_0\rho}{2\pi a^2} \hat{a}_\phi \dots\dots\dots(4.58) \end{aligned}$$

and in the region ,  $a \leq \rho \leq b$

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$$\vec{B}_e = \hat{a}_\phi \frac{\mu_0 I}{2\pi\rho} \dots\dots\dots(4.59)$$

Let us consider the flux linkage per unit length in the inner conductor. Flux enclosed between the region  $\rho$  and  $\rho+d\rho$  ( and unit length in the axial direction).

$$d\phi_i = \frac{\mu_0 I}{2\pi a^2} d\rho \dots\dots\dots(4.60)$$

Fraction of the total current it links is  $\frac{\rho^2}{a^2}$

$$\therefore d\Lambda_i = \frac{\mu_0 I}{2\pi} \frac{\rho^3}{a^4} d\rho$$

$$\Lambda_i = \int_0^a \frac{\mu_0 I}{2\pi} \frac{\rho^3}{a^4} d\rho = \frac{\mu_0 I}{8\pi} \dots\dots\dots(4.61)$$

Similarly for the region  $a \leq \rho \leq b$

$$d\phi_e = \frac{\mu_0 I}{2\pi\rho} d\rho = d\lambda_e \dots\dots\dots(4.62)$$

& 
$$\lambda_e = \frac{\mu_0 I}{2\pi} \int_a^b \frac{d\rho}{\rho} = \frac{\mu_0 I}{2\pi} \ln \frac{b}{a} \dots\dots\dots(4.63)$$

Total linkage  $\Lambda = \Lambda_i + \Lambda_e$

$$= \frac{\mu_0 I}{2\pi} \left[ \frac{1}{4} + \ln \left( \frac{b}{a} \right) \right] \dots\dots\dots(4.64)$$

The self inductance, 
$$L = \frac{\Lambda_i + \Lambda_e}{I} = \frac{\mu_0}{2\pi} \left[ \frac{1}{4} + \ln \left( \frac{b}{a} \right) \right] \dots\dots\dots(4.65)$$

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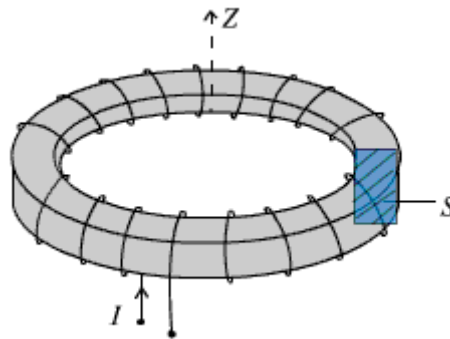
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$$\frac{\mu_0}{8\pi}$$

Here, the first term  $\frac{\mu_0}{8\pi}$  arises from the flux linkage internal to the solid inner conductor and is the internal inductance per unit length.

In high frequency application and assuming the conductivity to be very high, the current in the internal conductor instead of being distributed throughout remain essentially concentrated on the surface of the inner conductor ( as we shall see later) and the internal inductance becomes negligibly small.

**Example 3:** Inductance of an  $N$  turn toroid carrying a filamentary current  $I$ .



**Fig 4.12:**  $N$  turn toroid carrying filamentary current  $I$ .

**Solution:** Magnetic flux density inside the toroid is given by

$$\vec{B} = \frac{\mu I}{2\pi r} \hat{a}_\phi \dots\dots\dots(4.66)$$

Let the inner radius is 'a' and outer radius is 'b'. Let the cross section area 'S' is small compared to the mean

radius of the toroid  $r_0 \left[ = \frac{a+b}{2} \right]$

Then total flux

$$\phi = N \frac{\mu I}{2\pi r_0} S \dots\dots\dots(4.67)$$

and flux linkage

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$$\Lambda = \frac{\mu N^2 IS}{2\pi r_0} \dots\dots\dots(4.68)$$

The inductance

$$L = \frac{\Lambda}{I} = \frac{\mu N^2 S}{2\pi r_0} \dots\dots\dots(4.69)$$

**ENERGY STORED IN MAGNETIC FIELD:**

So far we have discussed the inductance in static forms. In earlier chapter we discussed the fact that work is required to be expended to assemble a group of charges and this work is stated as electric energy. In the same manner energy needs to be expended in sending currents through coils and it is stored as magnetic energy. Let us consider a scenario where we consider a coil in which the current is increased from 0 to a value *I*. As mentioned earlier, the self inductance of a coil in general can be written as

$$L = \frac{d\Lambda}{di} = N \frac{d\phi}{di} \dots\dots\dots(4.70a)$$

or  $L di = N d\phi \dots\dots\dots(4.70b)$

If we consider a time varying scenario,

$$L \frac{di}{dt} = N \frac{d\phi}{dt} \dots\dots\dots(4.71)$$

We will later see that  $N \frac{d\phi}{dt}$  is an induced voltage.

$\therefore v = L \frac{di}{dt}$  is the voltage drop that appears across the coil and thus voltage opposes the change of current.

Therefore in order to maintain the increase of current, the electric source must do an work against this induced voltage.

$$\begin{aligned} dW &= vi dt \\ &= Li di \end{aligned} \dots\dots\dots(4.72)$$

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$$W = \int_0^I Li \, di = \frac{1}{2} LI^2 \quad \text{(Joule)} \dots\dots\dots(4.73)$$

which is the energy stored in the magnetic circuit.

We can also express the energy stored in the coil in term of field quantities.

For linear magnetic circuit

$$W = \frac{1}{2} \frac{N\phi}{l} I^2 = \frac{1}{2} N\phi I \quad \dots\dots\dots(4.74)$$

Now,

$$\phi = \int_s \vec{B} \cdot d\vec{S} = BA \quad \dots\dots\dots(4.75)$$

where A is the area of cross section of the coil. If l is the length of the coil

$$NI = Hl$$

$$\therefore W = \frac{1}{2} HBAI \quad \dots\dots\dots(4.76)$$

AI is the volume of the coil. Therefore the magnetic energy density i.e., magnetic energy/unit volume is given by

$$W_m = \frac{W}{AI} = \frac{1}{2} BH \quad \dots\dots\dots(4.77)$$

In vector form

$$W_m = \frac{1}{2} \vec{B} \cdot \vec{H} \quad \text{J/mt}^3 \dots\dots\dots(4.78)$$

is the energy density in the magnetic field.