

Course/Branch : B.E./ECE	Year / Semester :II/IV	Format No.	NAC/TLP-07a.5
Subject Code :EC8451	Subject Name :ELECTROMAGNETIC FIELDS	Rev. No.	02
Unit No :4	Unit Name : TIME VARYING FIELDS AND MAXWELL'S EQUATIONS	Date	14-11-2017

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INTRODUCTION

In our study of static fields so far, we have observed that static electric fields are produced by electric charges, static magnetic fields are produced by charges in motion or by steady current. Further, static electric field is a conservative field and has no curl, the static magnetic field is continuous and its divergence is zero. The fundamental relationships for static electric fields among the field quantities can be summarized as:

$$\nabla \times \vec{E} = 0 \quad (5.1a)$$

$$\nabla \cdot \vec{D} = \rho_v \quad (5.1b)$$

For a linear and isotropic medium,

$$\vec{D} = \epsilon \vec{E} \quad (5.1c)$$

Similarly for the magnetostatic case

$$\nabla \cdot \vec{B} = 0 \quad (5.2a)$$

$$\nabla \times \vec{H} = \vec{J} \quad (5.2b)$$

$$\vec{B} = \mu \vec{H} \quad (5.2c)$$

It can be seen that for static case, the electric field vectors \vec{E} and \vec{D} and magnetic field vectors \vec{B} and \vec{H} form separate pairs.

In this chapter we will consider the time varying scenario. In the time varying case we will observe that a changing magnetic field will produce a changing electric field and vice versa.

We begin our discussion with Faraday's Law of electromagnetic induction and then present the Maxwell's equations which form the foundation for the electromagnetic theory.

FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

Michael Faraday, in 1831 discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In terms of fields, we can say that a time varying magnetic field produces an electromotive force (emf) which causes a current in a closed circuit. The quantitative

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relation between the induced emf (the voltage that arises from conductors moving in a magnetic field or from changing magnetic fields) and the rate of change of flux linkage developed based on experimental observation is known as Faraday's law. Mathematically, the induced emf can be written as

$$Emf = - \frac{d\phi}{dt} \text{ Volts} \quad (5.3)$$

where ϕ is the flux linkage over the closed path.

A non-zero $\frac{d\phi}{dt}$ may result due to any of the following:

- (a) time changing flux linkage a stationary closed path.
- (b) relative motion between a steady flux a closed path.
- (c) a combination of the above two cases.

The negative sign in equation (5.3) was introduced by Lenz in order to comply with the polarity of the induced emf. The negative sign implies that the induced emf will cause a current flow in the closed loop in such a direction so as to oppose the change in the linking magnetic flux which produces it. (It may be noted that as far as the induced emf is concerned, the closed path forming a loop does not necessarily have to be conductive).

If the closed path is in the form of N tightly wound turns of a coil, the change in the magnetic flux linking the coil induces an emf in each turn of the coil and total emf is the sum of the induced emfs of the individual turns, i.e.,

$$Emf = -N \frac{d\phi}{dt} \text{ Volts} \quad (5.4)$$

By defining the total flux linkage as

$$\lambda = N\phi \quad (5.5)$$

The emf can be written as

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$$Emf = - \frac{d\lambda}{dt} \quad (5.6)$$

Continuing with equation (5.3), over a closed contour 'C' we can write

$$Emf = \oint_C \vec{E} \cdot d\vec{l} \quad (5.7)$$

where \vec{E} is the induced electric field on the conductor to sustain the current.

Further, total flux enclosed by the contour 'C' is given by

$$\phi = \int_S \vec{B} \cdot d\vec{s} \quad (5.8)$$

Where S is the surface for which 'C' is the contour.

From (5.7) and using (5.8) in (5.3) we can write

$$\oint_C \vec{E} \cdot d\vec{l} = - \frac{\partial}{\partial t} \oint_S \vec{B} \cdot d\vec{s} \quad (5.9)$$

By applying stokes theorem

$$\int_S \nabla \times \vec{E} \cdot d\vec{s} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \quad (5.10)$$

Therefore, we can write

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (5.11)$$

which is the Faraday's law in the point form

$$\frac{d\phi}{dt}$$

We have said that non zero $\frac{d\phi}{dt}$ can be produced in a several ways. One particular case is when a time varying flux linking a stationary closed path induces an emf. The emf induced in a stationary closed path by a time varying magnetic field is called a **transformer emf**.

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EXAMPLE: IDEAL TRANSFORMER

As shown in figure 5.1, a transformer consists of two or more numbers of coils coupled magnetically through a common core. Let us consider an ideal transformer whose winding has zero resistance, the core having infinite permeability and magnetic losses are zero.

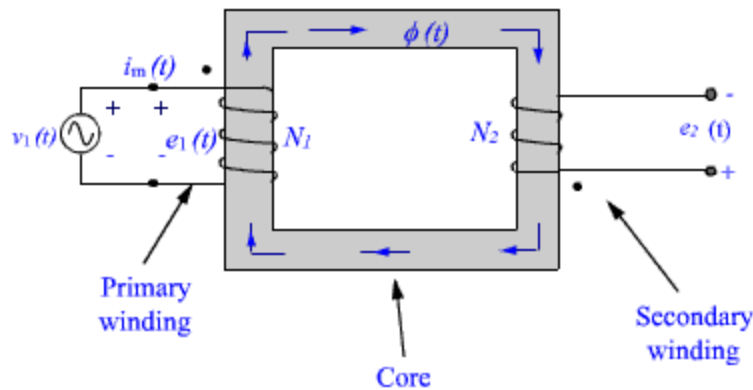


Fig 5.1: Transformer with secondary open

These assumptions ensure that the magnetization current under no load condition is vanishingly small and can be ignored. Further, all time varying flux produced by the primary winding will follow the magnetic path inside the core and link to the secondary coil without any leakage. If N_1 and N_2 are the number of turns in the primary and the secondary windings respectively, the induced emfs are

$$e_1 = N_1 \frac{d\phi}{dt} \tag{5.12a}$$

$$e_2 = N_2 \frac{d\phi}{dt} \tag{5.12b}$$

(The polarities are marked, hence negative sign is omitted. The induced emf is +ve at the dotted end of the winding.)

$$\therefore \frac{e_1}{e_2} = \frac{N_1}{N_2} \tag{5.13}$$

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i.e., the ratio of the induced emfs in primary and secondary is equal to the ratio of their turns. Under ideal condition, the induced emf in either winding is equal to their voltage rating.

$$\frac{v_1}{v_2} = \frac{N_1}{N_2} = a \quad (5.14)$$

where 'a' is the transformation ratio. When the secondary winding is connected to a load, the current flows in the secondary, which produces a flux opposing the original flux. The net flux in the core decreases and induced emf will tend to decrease from the no load value. This causes the primary current to increase to nullify the decrease in the flux and induced emf. The current continues to increase till the flux in the core and the induced emfs are restored to the no load values. Thus the source supplies power to the primary winding and the secondary winding delivers the power to the load. Equating the powers

$$i_1 v_1 = i_2 v_2 \quad (5.15)$$

$$\frac{i_2}{i_1} = \frac{v_1}{v_2} = \frac{e_1}{e_2} = \frac{N_1}{N_2} \quad (5.16)$$

Further,

$$i_2 N_2 - i_1 N_1 = 0 \quad (5.17)$$

i.e., the net magnetomotive force (mmf) needed to excite the transformer is zero under ideal condition.

MOTIONAL EMF:

Let us consider a conductor moving in a steady magnetic field as shown in the fig 5.2.

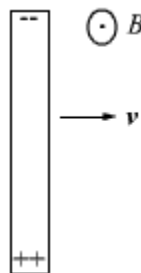


Fig 5.2

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If a charge Q moves in a magnetic field \vec{B} , it experiences a force

$$\vec{F} = Q\vec{v} \times \vec{B} \quad (5.18)$$

This force will cause the electrons in the conductor to drift towards one end and leave the other end positively charged, thus creating a field and charge separation continuous until electric and magnetic forces balance and an equilibrium is reached very quickly, the net force on the moving conductor is zero.

$\frac{\vec{F}}{Q} = \vec{v} \times \vec{B}$ can be interpreted as an induced electric field which is called the motional electric field

$$\vec{E}_m = \vec{v} \times \vec{B} \quad (5.19)$$

If the moving conductor is a part of the closed circuit C, the generated emf around the circuit is $\oint_C \vec{v} \times \vec{B} \cdot d\vec{l}$. This emf is called the **motional emf**.

MAXWELL'S EQUATION

Equation (5.1) and (5.2) gives the relationship among the field quantities in the static field. For time varying case, the relationship among the field vectors written as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (5.20a)$$

$$\nabla \times \vec{H} = \vec{J} \quad (5.20b)$$

$$\nabla \cdot \vec{D} = \rho \quad (5.20c)$$

$$\nabla \cdot \vec{B} = 0 \quad (5.20d)$$

In addition, from the principle of conservation of charges we get the equation of continuity

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad (5.21)$$

The equation 5.20 (a) - (d) must be consistent with equation (5.21).

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We observe that

$$\nabla \cdot \nabla \times \vec{H} = 0 = \nabla \cdot \vec{J} \quad (5.22)$$

Since $\nabla \cdot \nabla \times \vec{A}$ is zero for any vector \vec{A} .

Thus $\nabla \times \vec{H} = \vec{J}$ applies only for the static case i.e., for the scenario when $\frac{\partial \rho}{\partial t} = 0$.
A classic example for this is given below.

Suppose we are in the process of charging up a capacitor as shown in fig 5.3.

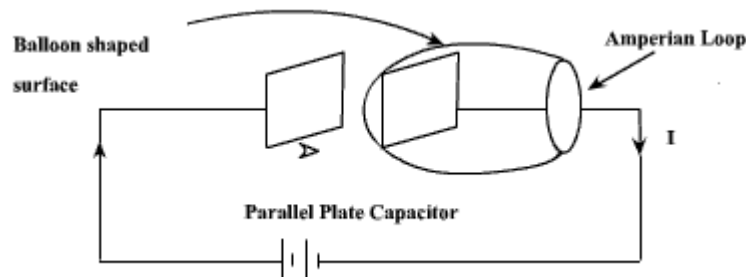


Fig 5.3

Let us apply the Ampere's Law for the Amperian loop shown in fig 5.3. $I_{enc} = I$ is the total current passing through the loop. But if we draw a balloon shaped surface as in fig 5.3, no current passes through this surface and hence $I_{enc} = 0$. But for non steady currents such as this one, the concept of current enclosed by a loop is ill-defined since it depends on what surface you use. In fact Ampere's Law should also hold true for time varying case as well, then comes the idea of displacement current which will be introduced in the next few slides.

We can write for time varying case,

$$\begin{aligned} \nabla \cdot (\nabla \times \vec{H}) &= 0 = \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} \\ &= \nabla \cdot \vec{J} + \frac{\partial}{\partial t} \nabla \cdot \vec{D} \\ &= \nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \end{aligned} \quad (5.23)$$

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$$\therefore \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (5.24)$$

The equation (5.24) is valid for static as well as for time varying case.

Equation (5.24) indicates that a time varying electric field will give rise to a magnetic field even in the absence of \vec{J} . The term $\frac{\partial \vec{D}}{\partial t}$ has a dimension of current densities (A/m^2) and is called the displacement current density.

Introduction of $\frac{\partial \vec{D}}{\partial t}$ in $\nabla \times \vec{H}$ equation is one of the major contributions of James Clerk Maxwell. The modified set of equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (5.25a)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (5.25b)$$

$$\nabla \cdot \vec{D} = \rho \quad (5.25c)$$

$$\nabla \cdot \vec{B} = 0 \quad (5.25d)$$

is known as the Maxwell's equation and this set of equations apply in the time varying scenario, static fields are being a particular case $\left(\frac{\partial}{\partial t} = 0\right)$.

In the integral form

$$\oint_c \vec{E} \cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \quad (5.26a)$$

$$\oint_c \vec{H} \cdot d\vec{l} = \int_s \left(J + \frac{\partial D}{\partial t} \right) \cdot d\vec{S} = I + \int_s \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S} \quad (5.26b)$$

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$$\int_V \nabla \cdot \vec{D} dv = \oint_S \vec{D} \cdot d\vec{S} = \int_V \rho dv \quad (5.26c)$$

$$\oint \vec{B} \cdot d\vec{S} = 0 \quad (5.26d)$$

The modification of Ampere's law by Maxwell has led to the development of a unified electromagnetic field theory. By introducing the displacement current term, Maxwell could predict the propagation of EM waves. Existence of EM waves was later demonstrated by Hertz experimentally which led to the new era of radio communication.

BOUNDARY CONDITIONS FOR ELECTROMAGNETIC FIELDS

The differential forms of Maxwell's equations are used to solve for the field vectors provided the field quantities are single valued, bounded and continuous. At the media boundaries, the field vectors are discontinuous and their behaviors across the boundaries are governed by boundary conditions. The integral equations (eqn 5.26) are assumed to hold for regions containing discontinuous media. Boundary conditions can be derived by applying the Maxwell's equations in the integral form to small regions at the interface of the two media. The procedure is similar to those used for obtaining boundary conditions for static electric fields (chapter 2) and static magnetic fields (chapter 4). The boundary conditions are summarized as follows;

With reference to fig 5.3

$$\hat{a}_n \times (\vec{E}_1 - \vec{E}_2) = 0 \quad 5.27(a)$$

$$\hat{a}_n \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s \quad 5.27(b)$$

$$\hat{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s \quad 5.27(c)$$

$$\hat{a}_n \cdot (\vec{B}_1 - \vec{B}_2) = 0 \quad 5.27(d)$$

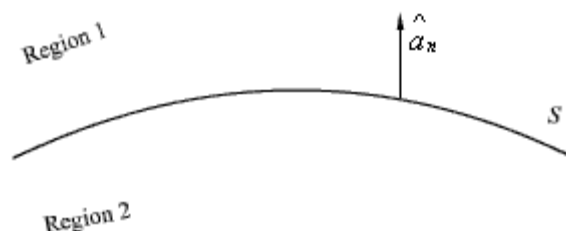


Fig 5.4

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Equation 5.27 (a) says that tangential component of electric field is continuous across the interface while from 5.27 (c) we note that tangential component of the magnetic field is discontinuous by an amount equal to the surface current density. Similarly 5.27 (b) states that normal component of electric flux density vector \vec{D} is discontinuous across the interface by an amount equal to the surface current density while normal component of the magnetic flux density is continuous. If one side of the interface, as shown in fig 5.4, is a perfect electric conductor, say region 2, a surface current \vec{J}_s can exist even though \vec{E} is zero as $\sigma = \infty$.

Thus eqn 5.27(a) and (c) reduces to

$$\hat{a}_n \times \vec{H} = \vec{J}_s \quad (5.28(a))$$

$$\hat{a}_n \times \vec{E} = 0 \quad (5.28(b))$$

WAVE EQUATION AND THEIR SOLUTION:

From equation 5.25 we can write the Maxwell's equations in the differential form as

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

Let us consider a source free uniform medium having dielectric constant ϵ , magnetic permeability μ and conductivity σ . The above set of equations can be written as

$$\nabla \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad (5.29(a))$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad (5.29(b))$$

$$\nabla \cdot \vec{E} = 0 \quad (5.29(c))$$

$$\nabla \cdot \vec{H} = 0 \quad (5.29(d))$$

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Using the vector identity ,

$$\nabla \times \nabla \times \vec{A} = \nabla \cdot (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

We can write from 5.29(b)

$$\begin{aligned} \nabla \times \nabla \times \vec{E} &= \nabla \cdot (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \\ &= -\nabla \times \left(\mu \frac{\partial \vec{H}}{\partial t} \right) \end{aligned}$$

or

$$\nabla \cdot (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \vec{H})$$

Substituting $\nabla \times \vec{H}$ from 5.29(a)

$$\nabla \cdot (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t} \left(\sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \right)$$

But in source free medium $\nabla \cdot \vec{E} = 0$ (eqn 5.29(c))

$$\nabla^2 \vec{E} = \mu \sigma \frac{\partial \vec{E}}{\partial t} + \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{5.30}$$

In the same manner for equation eqn 5.29(a)

$$\begin{aligned} \nabla \times \nabla \times \vec{H} &= \nabla \cdot (\nabla \cdot \vec{H}) - \nabla^2 \vec{H} \\ &= \sigma (\nabla \times \vec{E}) + \epsilon \frac{\partial}{\partial t} (\nabla \times \vec{E}) \\ &= \sigma \left(-\mu \frac{\partial \vec{H}}{\partial t} \right) + \epsilon \frac{\partial}{\partial t} \left(-\mu \frac{\partial \vec{H}}{\partial t} \right) \end{aligned}$$

Since $\nabla \cdot \vec{H} = 0$ from eqn 5.29(d), we can write

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$$\nabla^2 \vec{H} = \mu\sigma \left(\frac{\partial \vec{H}}{\partial t} \right) + \mu\epsilon \left(\frac{\partial^2 \vec{H}}{\partial t^2} \right) \quad (5.31)$$

These two equations

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \mu\sigma \left(\frac{\partial \vec{H}}{\partial t} \right) + \mu\epsilon \left(\frac{\partial^2 \vec{H}}{\partial t^2} \right)$$

are known as wave equations.

It may be noted that the field components are functions of both space and time. For example, if we consider a Cartesian co ordinate system, \vec{E} and \vec{H} essentially represents $\vec{E}(x,y,z,t)$ and $\vec{H}(x,y,z,t)$. For simplicity, we consider propagation in free space, i.e. $\sigma = 0$, $\mu = \mu_0$ and $\epsilon = \epsilon_0$. The wave eqn in equations 5.30 and 5.31 reduces to

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right) \quad (5.32(a))$$

$$\nabla^2 \vec{H} = \mu_0 \epsilon_0 \left(\frac{\partial^2 \vec{H}}{\partial t^2} \right) \quad (5.32(b))$$

Further simplifications can be made if we consider in Cartesian co ordinate system a special case where \vec{E} and \vec{H} are considered to be independent in two dimensions, say \vec{E} and \vec{H} are assumed to be independent of y and z. Such waves are called plane waves.

From eqn (5.32 (a)) we can write

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \epsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

The vector wave equation is equivalent to the three scalar equations

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$$\frac{\partial^2 \vec{E}_x}{\partial x^2} = \epsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E}_x}{\partial t^2} \right) \quad (5.33(a))$$

$$\frac{\partial^2 \vec{E}_y}{\partial x^2} = \epsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E}_y}{\partial t^2} \right) \quad (5.33(b))$$

$$\frac{\partial^2 \vec{E}_z}{\partial x^2} = \epsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E}_z}{\partial t^2} \right) \quad (5.33(c))$$

Since we have $\nabla \cdot \vec{E} = 0$,

$$\therefore \frac{\partial \vec{E}_x}{\partial x} + \frac{\partial \vec{E}_y}{\partial y} + \frac{\partial \vec{E}_z}{\partial z} = 0 \quad (5.34)$$

As we have assumed that the field components are independent of y and z eqn (5.34) reduces to

$$\frac{\partial E_x}{\partial x} = 0 \quad (5.35)$$

i.e. there is no variation of E_x in the x direction.

Further, from 5.33(a), we find that $\frac{\partial E_x}{\partial x} = 0$ implies $\frac{\partial^2 E_x}{\partial t^2} = 0$ which requires any three of the conditions to be satisfied: (i) $E_x=0$, (ii) $E_x = \text{constant}$, (iii) E_x increasing uniformly with time.

A field component satisfying either of the last two conditions (i.e (ii) and (iii)) is not a part of a plane wave motion and hence E_x is taken to be equal to zero. Therefore, a uniform plane wave propagating in x direction does not have a field component (E or H) acting along x.

Without loss of generality let us now consider a plane wave having E_y component only (Identical results can be obtained for E_z component) .

The equation involving such wave propagation is given by

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$$\frac{\partial^2 \vec{E}_y}{\partial x^2} = \epsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E}_y}{\partial t^2} \right) \quad (5.36)$$

The above equation has a solution of the form

$$E_y = f_1(x - v_0 t) + f_2(x + v_0 t) \quad (5.37)$$

where $v_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

Thus equation (5.37) satisfies wave eqn (5.36) can be verified by substitution. $f_1(x - v_0 t)$ corresponds to the wave traveling in the + x direction while $f_2(x + v_0 t)$ corresponds to a wave traveling in the -x direction. The general solution of the wave eqn thus consists of two waves, one traveling away from the source and other traveling back towards the source. In the absence of any reflection, the second form of the eqn (5.37) is zero and the solution can be written as

$$E_y = f_1(x - v_0 t) \quad (5.38)$$

Such a wave motion is graphically shown in fig 5.5 at two instances of time t_1 and t_2 .

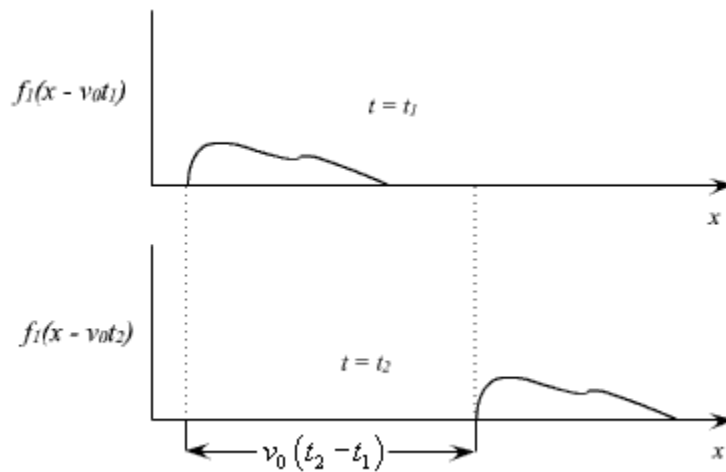


Fig 5.5 : Traveling wave in the + x direction

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Let us now consider the relationship between E and H components for the forward traveling wave.

Since $\vec{E} = \hat{a}_y E_y = \hat{a}_y f_1(x - v_0 t)$ and there is no variation along y and z.

$$\nabla \times \vec{E} = \hat{a}_z \frac{\partial E_y}{\partial x}$$

Since only z component of $\nabla \times \vec{E}$ exists, from (5.29(b))

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t} \quad (5.39)$$

and from (5.29(a)) with $\sigma = 0$, only H_z component of magnetic field being present

$$\nabla \times \vec{H} = -\hat{a}_y \frac{\partial H_z}{\partial x}$$

$$\therefore -\frac{\partial H_z}{\partial x} = \epsilon_0 \frac{\partial E_y}{\partial t} \quad (5.40)$$

Substituting E_y from (5.38)

$$\frac{\partial H_z}{\partial x} = -\epsilon_0 \frac{\partial E_y}{\partial t} = \epsilon_0 v_0 f_1'(x - v_0 t)$$

$$\therefore \frac{\partial H_z}{\partial x} = \epsilon_0 \frac{1}{\sqrt{\mu_0 \epsilon_0}} f_1'(x - v_0 t)$$

$$\therefore H_z = \sqrt{\frac{\epsilon_0}{\mu_0}} \int f_1'(x - v_0 t) dx + c$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}} \int \frac{\partial}{\partial x} f_1 dx + c$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}} f_1 + c$$

$$H_z = \sqrt{\frac{\epsilon_0}{\mu_0}} E_y + c$$

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The constant of integration means that a field independent of x may also exist. However, this field will not be a part of the wave motion.

$$\text{Hence } H_x = \sqrt{\frac{\epsilon_0}{\mu_0}} E_y \quad (5.41)$$

which relates the E and H components of the traveling wave.

$$z_0 = \frac{E_y}{H_x} = \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 120\pi \text{ or } 377\Omega$$

$z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$ is called the characteristic or intrinsic impedance of the free space.

TIME HARMONIC FIELDS:

So far, in discussing time varying electromagnetic fields, we have considered arbitrary time dependence. The time dependence of the field quantities depends on the source functions. One of the most important case of time varying electromagnetic field is the time harmonic (sinusoidal or co sinusoidal) time variation where the excitation of the source varies sinusoidally in time with a single frequency. For time-harmonic fields, phasor analysis can be applied to obtain single frequency steady state response. Since Maxwell's equations are linear differential equations, for source functions with arbitrary time dependence, electromagnetic fields can be determined by superposition. Periodic time functions can be expanded into Fourier series of harmonic sinusoidal components while transient non-periodic functions can be expressed as Fourier integrals. Field vectors that vary with space coordinates and are sinusoidal function of time can be represented in terms of vector phasors that depend on the space coordinates but not on time. For time harmonic case, the general time variation is $e^{j\omega t}$ and for a cosine reference, the instantaneous fields can be written as:

$$\vec{E}(x, y, z, t) = \text{Re} \left[\vec{E}(x, y, z) e^{j\omega t} \right] \quad (5.42)$$

where $\vec{E}(x, y, z)$ is a vector phasor that contain the information on direction, magnitude and phase. The phasors in general are complex quantities. All time harmonic field components can be written in this manner.

The time rate of change of \vec{E} can be written as:

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$$\frac{\partial \vec{E}(x, y, z, t)}{\partial t} = \text{Re} \left[j\omega \vec{E}(x, y, z) e^{j\omega t} \right] \quad (5.43)$$

Thus we find that if the electric field vector $\vec{E}(x, y, z, t)$ is represented in the phasor form as $\vec{E}(x, y, z)$, then $\frac{\partial \vec{E}(x, y, z, t)}{\partial t}$ can be represented by the phasor $j\omega \vec{E}(x, y, z)$. The integral $\int \vec{E}(x, y, z, t) dt$ can be represented by the phasor $\frac{\vec{E}(x, y, z)}{j\omega}$. In the same manner, higher order derivatives and integrals with respect to t can be represented by multiplication and division of the phasor $\vec{E}(x, y, z)$ by higher power of $j\omega$. Considering the field phasors (\vec{E}, \vec{H}) and source phasors (ρ, \vec{J}) in a simple linear isotropic medium, we can write the Maxwell's equations for time harmonic case in the phasor form as:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \quad (5.44a)$$

$$\nabla \times \vec{H} = \vec{J} + j\omega\epsilon\vec{E} \quad (5.44b)$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon} \quad (5.44c)$$

$$\nabla \cdot \vec{H} = 0 \quad (5.44d)$$

Similarly, the wave equations described in equation (5.32) can be written as:

$$\begin{aligned} \nabla^2 \vec{E} &= \mu_0\epsilon_0(j\omega)^2 \vec{E} = -\omega^2 \mu_0\epsilon_0 \vec{E} \\ \text{or } \nabla^2 \vec{E} + k_0^2 \vec{E} &= 0 \end{aligned} \quad (5.45a)$$

and in the same manner, for the magnetic field

$$\nabla^2 \vec{H} + k_0^2 \vec{H} = 0$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is called the **wave number**.