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Subject Code :EC8391	**Subject Name** : Control systems engineering	Rev. No.	02
Unit No : 05	**Unit Name : State variable analysis**	Date	14-11-2017

LECTURE NOTES

STATE VARIABLE ANALYSIS

TYPES:

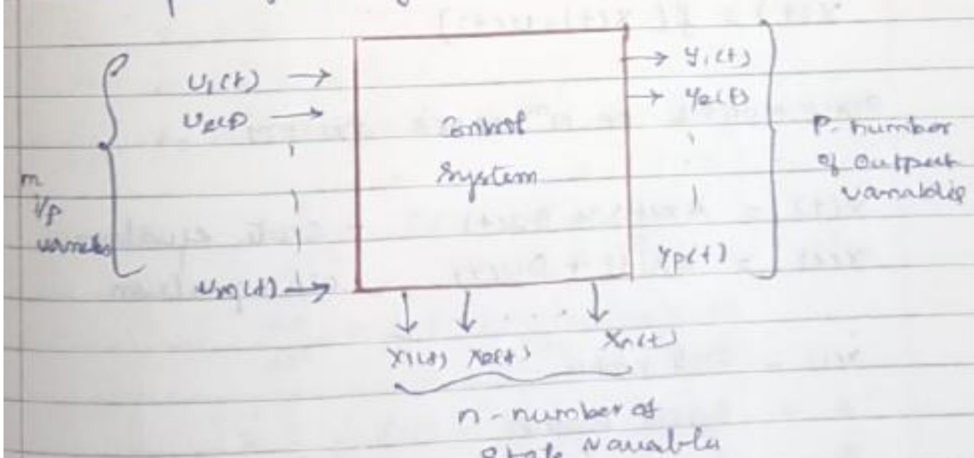
- Linear system
- Non linear system
- Time invariant system
- Time varying system
- Multiple input multiple output system

DRAWBACKS:

- Transfer function is defined under zero initial conditions
- It is applicable for linear time invariant systems
- Restricted for single input single output systems

STATE SPACE FORMULATION

STATE SPACE - The set of all possible values which the state vector X(t) can have at time 't' forms the state space of the system



- State is the condition of a system at any time instant, t
- A set of variable which describes the state of the system at any time instant are called state variable

LECTURE NOTES

STATE VECTOR – State vector is a $(n \times 1)$ column matrix whose elements are state variables.

Input vector $U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$

Output vector $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$

State variable vector $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

$$\dot{X}(t) = f(X(t), U(t))$$

STATE MODEL OF N^{th} ORDER SYSTEM

$\dot{X}(t) = AX(t) + BU(t)$ – State equation

$Y(t) = CX(t) + DU(t)$ – o/p equation

$\dot{X}(t)$ – State vector

A – System matrix

B – Input matrix

$Y(t)$ – Output vector

C – Output matrix

D – Transmission matrix

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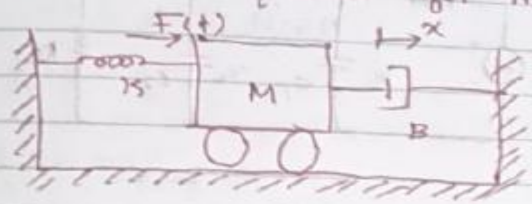
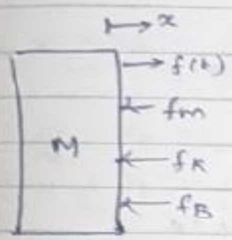
LECTURE NOTES

STATE SPACE REPRESENTATION – PHYSICAL VARIABLES

Write the state equation for mechanical system





$$f_m = M\frac{d^2x}{dt^2}$$

$$f_K = Kx$$

$$f_B = B.\frac{dx}{dt}$$

By Newton's law,

$$f_m + f_B + f_K = f(t)$$

$$M\frac{d^2x}{dt^2} + Kx + B.\frac{dx}{dt} = f(t) \rightarrow (1)$$

$$x_1 = x$$

$$x_2 = \dot{x}$$

$$\dot{x}_1 = \dot{x} = x_2$$

$$\dot{x}_2 = \ddot{x} = x_3$$

From (1),

$$M\frac{d^2x}{dt^2} = f(t) - Kx - B\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{1}{M}f(t) - \frac{K}{M}x = \frac{B}{M}\frac{dx}{dt}$$

$$\ddot{x} = \frac{1}{M}f(t) - \frac{K}{M}x_1 - \frac{B}{M}\dot{x}$$

$$\dot{x}_2 = \frac{1}{M}f(t) - \frac{K}{M}x_1 - \frac{B}{M}x_2$$

$$\dot{x}_1 = x_2$$

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\frac{K}{M} & -\frac{B}{M}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ 1/M\end{bmatrix}f(t)$$

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LECTURE NOTES

STATE SPACE REPRESENTATION – PHASE VARIABLES

Obtain the state model of the system whose transfer function is, $\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$$

$$Y(s)[s^3+4s^2+2s+1] = 10\,U(s)$$

$$s^3Y(s) + 4s^2Y(s) + 2sY(s) + Y(s) = 10\,U(s)$$

$$\dddot{y} + 4\ddot{y} + 2\dot{y} + y = 10u \quad \rightarrow (1)$$

State variables,

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}$$

$$\dddot{y} = \dot{x}_3$$
$$\ddot{y} = x_3$$
$$\dot{y} = x_2$$
$$y = x_1$$

$$\dot{x}_3 + 4x_3 + 2x_2 + x_1 = 10u$$

$$\dot{x}_3 = -x_1 - 2x_2 - 4x_3 + 10u.$$

The state equations are,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -x_1 - 2x_2 - 4x_3 + 10u$$

$$y = x_1$$

State model,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 10\end{bmatrix}[u]$$

$$Y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

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LECTURE NOTES

STATE SPACE REPRESENTATION - CANONICAL VARIABLES

Determine the canonical state model of the system the transfer function

$$\frac{Y(s)}{U(s)} = \frac{10(s+4)}{s(s+1)(s+2)}$$

$$\frac{10(s+4)}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

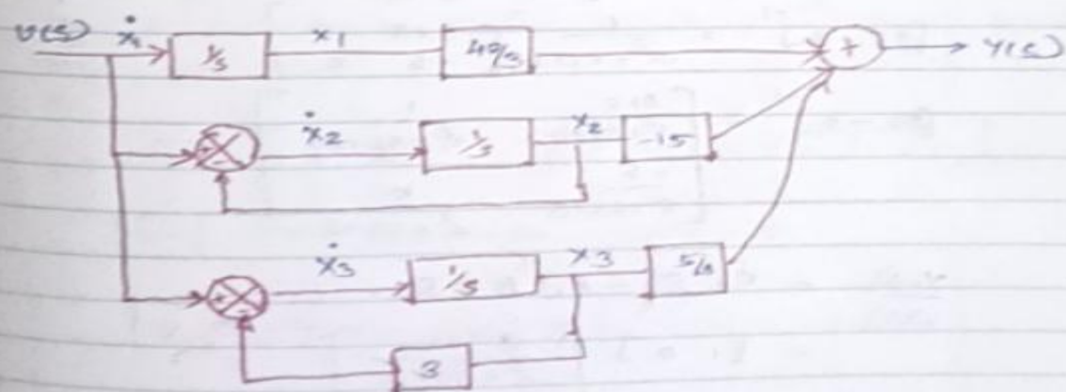
$$A = s \times \left.\frac{10(s+4)}{s(s+1)(s+3)}\right|_{s=0} \qquad A = 40/3$$

$$B = (s+1)\left.\frac{10(s+4)}{s(s+1)(s+3)}\right|_{s=-1} \qquad B = -15$$

$$C = (s+3) \times \left.\frac{10(s+4)}{s(s+1)(s+3)}\right|_{s=-3} \qquad C = 5/3$$

$$\frac{Y(s)}{U(s)} = \frac{40/3}{s} - \frac{15}{s+1} + \frac{5/3}{s+3}$$

$$= \frac{40/3}{s} - \frac{15}{s(1+1/s)} + \frac{5/3}{s(1+3/s)}$$



$\dot{x}_1 = u$

$\dot{x}_2 = -x_2 + u$

$\dot{x}_3 = -3x_3 + u$

O/P equation, $y = \frac{40}{3}x_1 - 15x_2 + 5/3\, x_3$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}[u]$$

$$y = \begin{bmatrix}40/3 & -15 & 5/3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

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LECTURE NOTES

TRANSFER FUNCTION - STATE VARIABLE REPRESEN

Obtain transfer function model for following state model system

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}\quad D = [0]$$

$$TF,\ \frac{Y(s)}{U(s)} = C[sI - A]^{-1}B + D$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{adj[sI - A]}{|sI - A|}$$

$$= \begin{vmatrix} s & -1 \\ 6 & s+5 \end{vmatrix} = s(s+5) + 6 = s^2 + 5s + 6$$

$$adj[sI - A] = \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{s^2 + 5s + 6}\begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$[sI - A]^{-1} = \begin{bmatrix} \frac{s+5}{s^2+5s+6} & \frac{1}{s^2+5s+6} \\ \frac{-6}{s^2+5s+6} & \frac{s}{s^2+5s+6} \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = C[sI - A]^{-1}B + D$$

$$= [1 \quad 0]\begin{bmatrix} \frac{s+5}{s^2+5s+6} \\ \frac{-6}{s^2+5s+6} \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{s+5}{s^2+5s+6}$$

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LECTURE NOTES

CONTROLLABILITY & OBSERVABILITY

Controllability:

A system is said to be completely state controllable if it is possible to transfer the system state from any initial state $X(t_0)$ to any other desired state $X(t_d)$ in finite time interval

$$Q_c = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

n - no of state variables

$|Q_c| \neq 0$ controllable, rank = n

$|Q_c| = 0$, controllable, rank < n

Observability

A system is said to be completely observable if every state $X(t)$ can be completely identified by measurements of the output $y(t)$ over a finite time interval

$$Q_o = [C^T \quad A^T C^T \quad (A^T)^2 C^T \quad \dots \quad (A^T)^{n-1} C^T]$$

$|Q_o| \neq 0$, observable rank = n

$|Q_o| = 0$, not observable, rank < n

LECTURE NOTES

A system is characterized by $\frac{Y(s)}{U(s)} = \frac{3}{s^3+5s^2+11s+6}$. Identify the first state as output. Determine whether or not the system is completely controllable and observable.

$$\frac{Y(s)}{U(s)} = \frac{3}{s^3+5s^2+11s+6}$$

$$Y(s)\left[s^3+5s^2+11s+6\right] = 3\left[U(s)\right]$$

$$s^3Y(s) + 5s^2Y(s) + 11sY(s) + 6Y(s) = 3U(s)$$

$$\dddot{y} + 5\ddot{y} + 11\dot{y} + 6y = 3u$$

State variables,

$$x_1 = y$$
$$x_2 = \dot{y} = \dot{x}_1$$
$$\dot{x}_2 = \ddot{y} = x_3$$
$$\dot{x}_3 = \dddot{y}$$
$$\dot{x}_3 + 5x_3 + 11x_2 + 6x_1 = 3u$$
$$\dot{x}_3 = 3u - 5x_3 - 11x_2 - 6x_1$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_1 = x_2$$

State matrix.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} u$$

$$y = x_1.$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

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LECTURE NOTES

Controllability $Q_c = [B \quad AB \quad A^2B]$

$$B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -15 \end{bmatrix}$$

$$A(AB) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ -15 \end{bmatrix} = \begin{bmatrix} 3 \\ -15 \\ 45 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & -15 \\ 3 & -15 & 45 \end{bmatrix}$$

$$|Q_c| = 3(-9) = -27 \neq 0$$

The system is controllable

Observability $Q_o = [C^T \quad A^T C^T \quad (A^T)^2 C^T]$

$$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T(A^T C^T) = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q_o = 1 \neq 0$$

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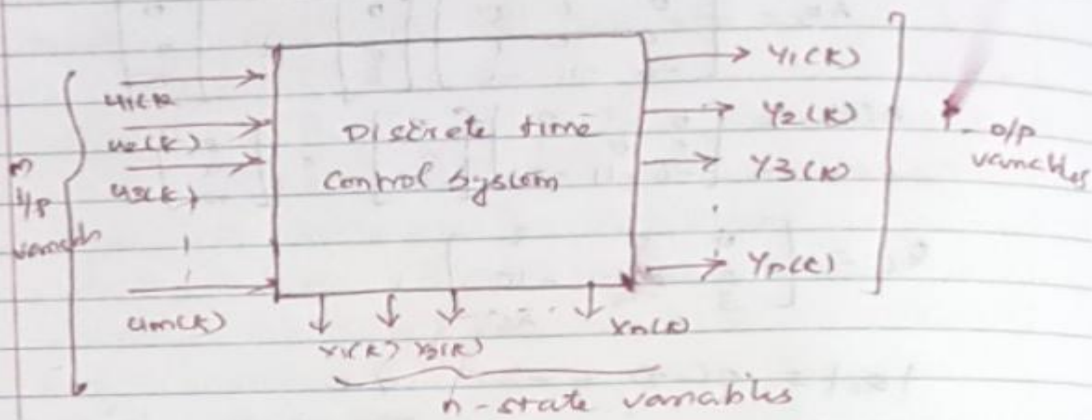
LECTURE NOTES

STATE SPACE REPRESENTATION OF DISCRETE TIME SYSTEM



state : $X(k+1) = A(X(k)) + B\,U(k)$

o/p : $Y(k) = C\,X(k) + D\,U(k)$

A discrete time system has transfer function

$$\frac{Y(z)}{U(z)} = \frac{4z^3 - 12z^2 + 13z - 7}{(z-1)^2(z-2)}$$

Determine the state model of the system in (a) phase variable form (b) Jordan canonical form.

(a) Phase variable form

$$\frac{Y(z)}{U(z)} = \frac{4z^3 - 12z^2 + 13z - 7}{(z-1)^2(z-2)}$$

$$= \frac{4z^3 - 12z^2 + 13z - 7}{(z^2 - 2z + 1)(z-2)}$$

$$= \frac{4z^3 - 12z^2 + 13z - 7}{z^3 - 2z^2 + z - 2z^2 + 4z - 2}$$

$$= \frac{4z^3 - 12z^2 + 13z - 7}{z^3 - 4z^2 + 5z - 2}$$

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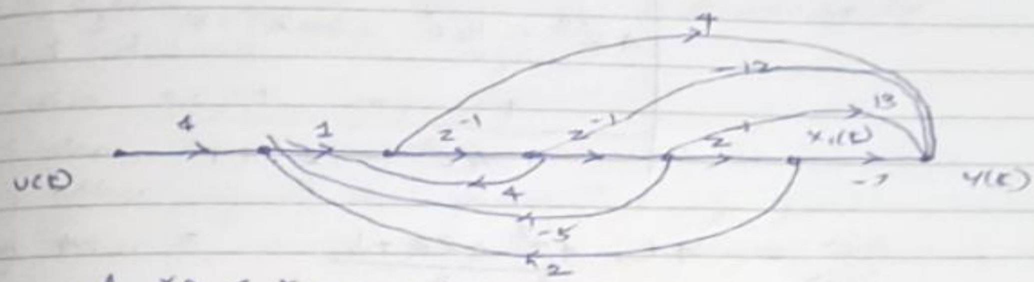
LECTURE NOTES

$$= \frac{4 - 12z^{-1} - 13z^{-2} - 12^{-3}}{1 - 4z^{-1} + 5z^{-2} - 2z^{-3}}$$



From node x_2 & x_3 we get.

$x_1(k+1) = x_2$

$x_2(k+1) = x_3$

$x_3(k+1) = 2x_1 - 5x_2 + 4x_3 + u$

State equation

$y = -7x_1 + 13x_2 - 12x_3 + 4x_3(k+1)$

$y = x_1 - 7x_2 + 4x_3 + u \longrightarrow$ o/p equation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [4]u$$

Jordan canonical form

$$\frac{Y(z)}{U(z)} = \frac{4z^3 - 12z^2 + 13z - 7}{(z-1)^2(z-2)}$$

$$= \frac{4z^3 - 12z^2 + 13z - 7}{z^3 - 4z^2 + 5z - 2}$$

LECTURE NOTES

$$z^3-4z^2+5z-2 \;\Big)\; 4z^3-10z^2+13z-7 \;\Big(\; 4$$

$$4z^3-16z^2+20z-8$$

$$-\quad +\quad -\quad +$$

$$4z^2-7z+1$$

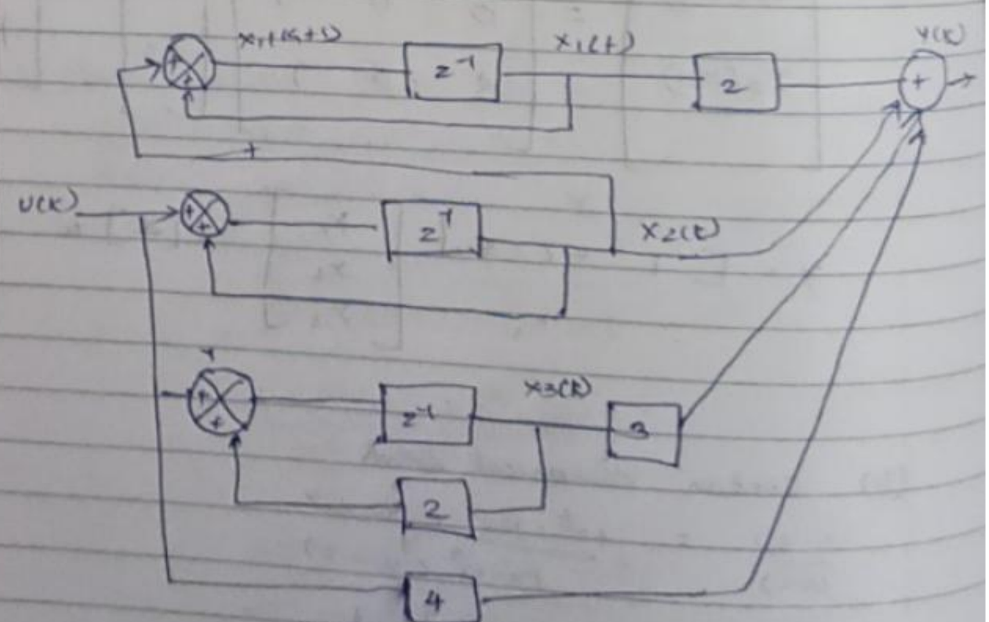
$$\frac{Y(z)}{U(z)} = 4 + \frac{4z^2-7z+1}{z^3-4z^2+5z-2} = 4 + \frac{4z^2-7z+1}{(z-1)^2(z-2)}$$

By partial fraction,

$$\frac{Y(z)}{U(z)} = 4 + \frac{A}{(z-1)^2} + \frac{B}{(z-1)} + \frac{C}{(z-2)}$$

$A = 2$; $B = 1$; $C = 3$

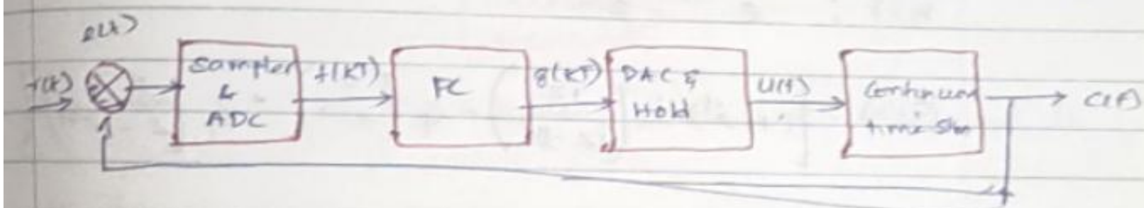
$$\frac{Y(z)}{U(z)} = 4 + \frac{2}{(z-1)^2} + \frac{1}{(z-1)} + \frac{3}{(z-2)}$$



SAMPLED DATA CONTROL SYSTEM

When the signal or information at any point in the system is in form of discrete pulses, it is called discrete data system.

1) when PC is employed
2) when control components are used in time basis
3) when signals are transmitted by pulse modulation.
4) when o/p or i/p component is digital / discrete signal.

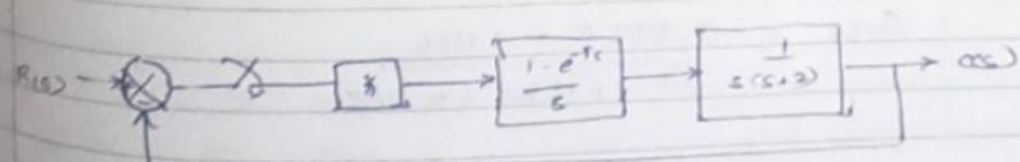


SAMPLING THEOREM :

A band limited continuous time signal with highest frequency fm Hertz, can be uniquely recovered from its samples provided that the sampling rate Fs is greater than or equal to 2fm samples per second.

$$F_s \geq 2f_m$$

A sampled data control system is shown in figure. Find the open loop pulse transfer function if the controller gain is unity with sampling time 0.5 sec.



$$G(z) = z\left[K\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{1}{s(s+2)}\right)\right]$$

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$$G(s) = z\left[k(1-e^{-Ts})\left(\frac{1}{s^2(s+2)}\right)\right]$$

$$\frac{1}{s^2(s+2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2}$$

$$A = \frac{1}{2} \qquad B = \frac{-1}{4} \qquad C = \frac{1}{4}.$$

$$G(z) = z\left[k(1-e^{-Ts})\left(\frac{1/2}{s^2}\right) - \left(\frac{1}{4}\right)\left(\frac{1}{s}\right) + \frac{1}{4}\left(\frac{1}{s+2}\right)\right]$$

$$k = 1 ; \quad e^{-Ts} = z^{-1}$$

$$G(z) = \left[(1-z^{-1})\,\frac{1}{2}\left(\frac{Tz}{(z-1)^2}\right) - \frac{1}{4}\left(\frac{z}{z-1}\right) + \frac{1}{4}\left(\frac{z}{z-e^{-2T}}\right)\right]$$

$$= \left(\frac{z-1}{z}\right)\left[\frac{1}{2}\left(\frac{Tz}{(z-1)^2}\right) - \frac{1}{4}\left(\frac{z}{z-1}\right) + \frac{1}{4}\left(\frac{z}{z-e^{-2T}}\right)\right]$$

$$G(z) = \frac{1}{2}\left(\frac{T}{(z-1)}\right) - \frac{1}{4} + \frac{1}{4}\left(\frac{z-1}{z-e^{-2T}}\right)$$

$$T = 0.5\ sec$$

$$G(z) = \frac{0.25}{z-1} - \frac{1}{4} + \frac{1}{4}\left(\frac{z-1}{z-e^{-1}}\right)$$

$$= \frac{0.25}{z-1} - \frac{1}{4} + \frac{1}{4}\left(\frac{z-1}{z-0.368}\right)$$

$$G(z) = \frac{0.0922 + 0.066}{z^2 - 1.3682 + 0.368}$$