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Subject Code :MA8251	**Subject Name** :ENGINEERING MATHEMATICS-II	Rev. No.	02
Unit No : 02	**Unit Name** : **VECTOR CALCULUS**	Date	14-11-2017

LECTURE NOTES

2.VECTOR CALCULUS

Scalar point function:

Let R be a region of space at each point of which a scalar $\emptyset = \emptyset(x, y, z)$ is given then $\emptyset$ is called a scalar point function.

Vector point function:

Let R be a region of space at each point of which a vector $\vec{v} = \vec{v}(x, y, z)$ is given then $\vec{v}$ is called a vector point function.

Gradient of a Scalar function:

Grad $\emptyset = \nabla\emptyset = \vec{i}\frac{\partial\emptyset}{\partial x} + \vec{j}\frac{\partial\emptyset}{\partial y} + \vec{k}\frac{\partial\emptyset}{\partial z}$

Note: $\nabla\emptyset \neq \emptyset\nabla$

Directional derivative:

The directional derivative of a scalar point $\emptyset = \emptyset(x, y, z)$ in the direction of $\vec{a}$ at any point P is given

$$\frac{d\emptyset}{ds} = \frac{\nabla\emptyset.\vec{a}}{|\vec{a}|}.$$

Maximum Directional derivative:

The Maximum Directional derivative is $|\nabla\emptyset|$ or $|\text{grad }\emptyset|$.

Unit normal vector:

Unit normal vector to the surface $\emptyset(x, y, z) = C$ is given by

Angle between the Surfaces:

The angle between the surfaces at a point is the angle between the normal to the surfaces at the point.

$$cos\theta = \frac{\nabla\emptyset_1.\nabla\emptyset_2}{|\nabla\emptyset_1|.|\nabla\emptyset_2|}$$

Orthogonal Surfaces:

Two Surfaces $\emptyset_1$ and $\emptyset_2$ form orthogonal families if

$\nabla\emptyset_1.\nabla\emptyset_2 = 0.$

Divergence of a vector function:

The Divergence of a vector $\vec{F}$ is

$$div\ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Solenoidal Vector:

A vector $\vec{F}$ is called Solenoidal Vector if $div\ \vec{F} = 0.$

(i.e) $\nabla.\vec{F}= 0$.

Curl of a vector function:

If rotation of $\vec{F}$ is defined as curl $\vec{F} = \nabla X \vec{F}$

Irrotational vector:

If $\vec{F}$ is a vector such that $\nabla X \vec{F} = 0$ at all the points in a given region. Then it is said eo be irrational vector in that region.

Workdone by a force:

Workdone by the force $\vec{F} = \int_c \vec{F}.\overrightarrow{dr}$

Conservative Force Field:

$\vec{F}$ is a conservative field. If $\int_A^B \vec{F}.\overrightarrow{dr}$ is independent of the path joining A and B.

$\int_A^B \vec{F}.\overrightarrow{dr} = \int_A^B d\emptyset = \emptyset(B) - \emptyset(A)$.

Integral theorems:

The following theorems called integral theorems

(i). Green's theorem

(ii). Stoke's theorem

(iii).Gauss divergence theorem

Green's theorem in a plane:

If C is a simple closed curve enclosing a region R. in the xy-plane and P(X,Y) , Q(X,Y) and its first order partial derivative are continuous in R then

$$\int_c P.dx + Q.dy = \iint_R \left(\frac{\partial Q}{\partial X} - \frac{\partial P}{\partial y}\right)$$

Stoke's theorem:

If $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve C, then

$\int_c \vec{F}.\overrightarrow{dr} = \iint_S Curl\,\vec{F}\,.\vec{n}.\overrightarrow{ds}$. where $\vec{n}$ in the unit normal vector at any point of S.

Gauss divergence theorem:

A vector function $\vec{F}$ taken over a closed surface S enclosing a volume V is equal to the volume integral of the divergence of $\vec{F}$ taken throughtout the volume V.

$$\iint_S \vec{F}\,.\vec{n}.\overrightarrow{ds} = \iiint_V \nabla.\vec{F}.dv$$

Problem:1

Verify Gauss divergence theorem for

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$$

Taken over the rectangular parallelopiped

$$0 \le x \le a, 0 \le y \le\le b, 0 \le z \le c.$$

Solu: By Gauss divergence theorem

$$= \iiint_V \nabla.\vec{F}.dv$$

Give $\iint_S \vec{F}.\vec{n}.\overrightarrow{ds}$n $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$

RHS $= \iiint_V \nabla.\vec{F}.dv$

$$\nabla.\vec{F} = \frac{\partial(x^2-yz)}{\partial x} + \frac{\partial(y^2-xz)}{\partial y} + \frac{\partial(z^2-xy)}{\partial z}$$

$$= 2x + 2y + 2z = 2(x + y + z)$$

$$\iiint_V \nabla.\vec{F}.dv = \int_0^c \int_0^b \int_0^a 2(x + y + z)dx\,dy\,dz$$

$$= 2\int_0^c \int_0^b \left(\frac{x^2}{2} + yx + zx\right)_0^a dy\,dz$$

$$= 2\int_0^c \int_0^b \left(\frac{a^2}{2} + ya + za\right)_0^a dy\,dz$$

$$= 2\int_0^c \left(\frac{a^2y}{2} + \frac{y^2}{2}a + zya\right)_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2b}{2} + \frac{b^2a}{2} + zba\right)_0^b dz$$

$$= 2\left(\frac{a^2bz}{2} + \frac{ab^2z}{2} + \frac{z^2}{2}ba\right)_0^c$$

$$= 2\left(\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right)$$

$$= 2\left(\frac{abc}{2}\right)(a + b + c)$$

LHS

To find $\iint_S \vec{F}.\vec{n}.\overrightarrow{ds}$

$$\iint_S \vec{F}.\vec{n}.\overrightarrow{ds} = \iint_{S_1} \vec{F}.\vec{n}.\overrightarrow{ds} + \iint_{S_2} \vec{F}.\vec{n}.\overrightarrow{ds} + \iint_{S_3} \vec{F}.\vec{n}.\overrightarrow{ds} + \iint_{S_4} \vec{F}.\vec{n}.\overrightarrow{ds} + \iint_{S_5} \vec{F}.\vec{n}.\overrightarrow{ds} + \iint_{S_6} \vec{F}.\vec{n}.\overrightarrow{ds}$$

Surface	$\vec{n}$	Equation of the surface	$\vec{F}.\vec{n}$	ds
S_1	$\vec{i}$	X=a	$(x^2 - yz)$	$dydz$
S_2	$\overrightarrow{-i}$	X=0	$-(x^2 - yz)$	$dydz$
S_3	$\vec{j}$	Y=b	$(y - xz)$	$dxdz$
S_4	$-\vec{j}$	Y=0	$(y^2 - xz)$	$dxdz$
S_5	$\vec{k}$	Z=c	$(z^2 - xy)$	$dxdy$
S_6	$\overrightarrow{-k}$	Z=0	$(z^2 - xy)$	$dxdy$

$$\iint_{S_1} \vec{F}\,.\vec{n}.\,\overrightarrow{ds} = \int_0^c \int_0^b (a^2 - yz)\,dydz$$

$$= a^2bc - \frac{b^2c^2}{4}$$

$$\iint_{S_2} \vec{F}\,.\vec{n}.\,\overrightarrow{ds} = -\int_0^c \int_0^b (0 - yz)\,dydz = \frac{c^2b^2}{4}$$

$$\iint_{S_3} \vec{F}\,.\vec{n}.\,\overrightarrow{ds} = \int_0^c \int_0^a (b^2 - xz)\,dxdz$$

$$= ab^2c - \frac{a^2c^2}{4}$$

$$\iint_{S_4} \vec{F}\,.\vec{n}.\,\overrightarrow{ds} = -\int_0^c \int_0^a (0 - xz)\,dxdz$$

$$= \frac{a^2c^2}{4}$$

$$\iint_{S_5} \vec{F}\,.\vec{n}.\,\overrightarrow{ds} = \int_0^b \int_0^a (z^2 - xy)\,dxdy$$

$$= abc^2 - \frac{a^2b^2}{4}$$

$$\iint_{S_6} \vec{F}\,.\vec{n}.\,\overrightarrow{ds} = -\int_0^b \int_0^a (0 - xy)\,dxdy$$

$$= \frac{a^2b^2}{4}$$

$$\iint_S \vec{F}\,.\vec{n}.\,\overrightarrow{ds} = a^2bc - \frac{b^2c^2}{4} + \frac{c^2b^2}{4} + ab^2c - \frac{a^2c^2}{4} + \frac{a^2c^2}{4}$$

$$+abc^2 - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

$$= a^2bc + ab^2c + abc^2$$

$= abc(a+b+c)$

LHS =RHS

Hence Gauss divergence theorem verified.

Problem:2

Verify stoke's theorem for $\vec{F} = (x^2+y^2)\vec{i} - (2xy)\vec{j}$ taken round the rectangle bounded by $x = \pm a, y = 0, y = b.$

Solu:

By stoke's theorem

$\int_c \vec{F}.\overrightarrow{dr} = \iint_S Curl\,\vec{F}.\vec{n}.\overrightarrow{ds}$

$\vec{F} = (x^2+y^2)\vec{i} - (2xy)\vec{j}$

$$Curl\,\vec{F} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & 2xy & 0 \end{pmatrix}$$

$Curl\,\vec{F} = -4y\vec{k}$

$\iint_S Curl\,\vec{F}.\vec{n}.\overrightarrow{ds} = \int_0^b \int_{-a}^{a} (-4y)\,dxdy$

$-4\int_0^b (x)_{-a}^{a}\, y$

$= -4ab^2$

$\int_c \vec{F}.\overrightarrow{dr} = \int_{c_1} \vec{F}.\overrightarrow{dr} + \int_{c_2} \vec{F}.\overrightarrow{dr} + \int_{c_3} \vec{F}.\overrightarrow{dr} + \int_{c_4} \vec{F}.\overrightarrow{dr}$

$$\int_{c_1} \vec{F}.\overrightarrow{dr} = \int_{-a}^{a}(x^2+y^2)dx-(2xy)\text{dy} = \int_{-a}^{a}x^2dx = \frac{2a^3}{3}$$

$$\int_{c_2} \vec{F}.\overrightarrow{dr} = \int_{0}^{b}(x^2+y^2)dx-(2xy)\text{dy} = -ab^2$$

$$\int_{c_3} \vec{F}.\overrightarrow{dr} = \int_{a}^{=a}(x^2+y^2)dx-(2xy)\text{dy} = -\frac{2a^3}{3}-2ab^2$$

$$\int_{c_4} \vec{F}.\overrightarrow{dr} = \int_{b}^{0}(x^2+y^2)dx-(2xy)\text{dy} = -ab^2$$

$$\int_{c} \vec{F}.\overrightarrow{dr} = \frac{2a^3}{3}-ab^2-\frac{2a^3}{3}-2ab^2-ab^2$$

$$= -4ab^2$$

LHS = RHS

Hence Stoke's theorem verified.