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Subject Code :MA8251	**Subject Name** :ENGINEERING MATHEMATICS-II	Rev. No.	02
Unit No : 03	**Unit Name** : **Analytic Functions**	Date	14-11-2017

LECTURE NOTES

3. Analytic Functions:

A single valued function $w = f(z)$ of a complex variable z is said to be analytic at a point $z_{0.}$ If it has a unique derivative at $z_{0.}$.

Cauchy-Riemann Equations :

Sufficient condition for $w = f(z)$ to be analytic

If the four partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ should exists and continuous.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Then only $w = f(z)$ is analytic.

Harmonic functions:

Any function of x and y which possesses continuous first and second order partial derivatives and satisfies Laplace's equation is called harmonic function

if u is harmonic

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0.$$

if V is harmonic

$$\frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 v}{\partial^2 y} = 0$$

Properties of Analytic functions;

1. Both real and imaginary parts of an analytic function are harmonic (or) satisfies laplace equation.

2. If $f(z) = u(x, y) + iv(x, y)$ is an analytic function then the $u(x, y) = c_1$ and $V(x, y) = c_2$ where c_1 and c_2 are constants are orthogonal to each other.
3. An analytic function with constant modulus is constant.

4. If $f(z)$ and f($\bar{z}$) are analytic function of z. then $f(z)$ is constant.

Construction of an analytic function:(Milne's thomsan method)

If $f(z) = u + iv$ is an analytic function

(i). When the real part u is given

Then $f(z) = \int u_x(z, 0)dz - iu_y(z, 0)dz + c$

(ii). When the imaginary part V is given

Then $f(z) = \int V_y(z, 0)dz + i\, V_X(z, 0)dz + c$

Conformal mapping:

A mapping $w = f(z)$ is said to be conformal at $z = z_0$ .If preserves the angle between any two curves through z_0 in z plane both in magnitude and direction.

Critical point :

A point z at which $f'(z) = 0$ is called critical point of the transformation $w = f(z)$.

Fixed point:

Fixed point (or) invariant point of a mapping $w = f(z)$ are points that are mapped onto themselves.

Note: Put W=Z in $w = f(z)$ and solve for Z.

The transformation: $w = z + k$(Translation)

Let $w = z + k$ where k is a complex constant.

$u + iv = x + iy + a + ib$

Where $k = a + ib$

$$u = x + +a \,; v = y + b$$

Therefore any point (x, y) in the Z-plane is mapped onto the point

$(x + a, y + b)$ in the W-plane.

The transformation: $w = kz$(magnification and rotation)

Consider $w = f(z)$

Where k,z,w are complex numbers.

$u + iv = k(x + iy)$

$u = kx, \ \ v = ky$

$x = \frac{u}{k}, \ \ y = \frac{v}{k}$

Thus any region in z-plane is mapped to a region with a rotation and magnification in the w-plane.

The Transfromation $w = \frac{1}{z}$ (Inversion and Reflection)

Given $w = \frac{1}{z}$

$z = \frac{1}{w}$

i.e $x + iy = \frac{1}{u+iv}$

$= \frac{u-iv}{(u+iv)(u-iv)}$

$x + iy = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$

$x + iy = = \frac{u}{u^2+v^2} + \frac{-iv}{u^2+v^2}$

Equating real and imaginary parts

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

Then $x^2 + y^2 = \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}$

$$x^2 + y^2 = \frac{u^2+v^2}{(u^2+v^2)^2} = \frac{1}{u^2+v^2}$$

Bilinear Transformation:

The transformation $w = \frac{az+b}{cz+d}$ where a,b,c,d are complex constants and ad-bc$\neq$ 0 is called the bilinear transformation.

Cross ratio:

if z_1, z_2, z_3, z_4 are four complex numbers, then

$$\frac{(z_1 - z_2)}{(z_1 - z_4)} \frac{(z_3 - z_4)}{(z_3 - z_2)}$$

is called the cross ratio of four points z_1, z_2, z_3, z_4 .

Bilinear transformation that maps the points z_1, z_2, z_3, z_4 of z-plane into the points w_1, w_2, w_3, w_4, of w-plane is given by

$$\frac{(w - w_1)}{(w - w_3)} \frac{(w_2 - w_3)}{(w_2 - w_1)} = \frac{(z - z_1)}{(z - z_3)} \frac{(z_2 - z_3)}{(z_2 - z_1)}$$

Properties of bilinear transformation:

1. The bilinear transformation always transforms circles into circles with Lines as limiting cases.

2. The bilinear transformation preserves cross ratio of cross ratio of four points

Problem:1

Find the analytic function of $u = \frac{sin2x}{cosh2y+cos2x}$. Find the corresponding analytic function $f(z) = u + iv$.

Solu:

Given $u = \frac{sin2x}{cosh2y+cos2x}$

By Milne's method

$$f(z) = \int u_x(z,0)dz - iu_y(z,0)dz + c$$

$$u_x = \frac{(cosh2y+cos2x)2\,cos2x - sin2x(-2sin2x)}{(cosh2y+cos2x)^2}$$

Put x=z, y=0.

$u_x(z,0)$

$$= \frac{(1+cos2z)2\,cos2z - sin2z(-2sin2z)}{(cosh0+cos2z)^2}$$

$$= \frac{2(1+cos2z)}{(1+cos2z)^2}$$

$$= \frac{2}{cos^2 z}$$

$$= sec^2 z$$

$$u_y = \frac{(cosh2y+cos2x)0 - sin2x(2sinh2y)}{(cosh2y+cos2x)^2}$$

Put x=z, y=0.

$$u_y(z,0) = 0$$

$$f(z) = \int u_x(z,0)dz - iu_y(z,0)dz + c$$

$$f(z) = \int sec^2 z\, dz + c$$

$$f(z) = tanz + C$$

Problem:2

Find the image of the circle $|z| = 2$ under the transformation

$W = Z + 3 + 2i.$

Solu:

Given $W = Z + 3 + 2i.$

$u + iv = x + iy + 3 + 2i.$

$u + iv = x + 3 + i(y + 2).$

Equating real and imaginary parts

$u = x + 3$; $v = y + 2$;

$x = u - 3$; $y = v - 2$

Given $|z| = 2$

$|x + iy| = 2$

$\sqrt{x^2 + y^2} = 2$

$x^2 + y^2 = 4$

$(u - 3)^2 + (v - 2)^2 = 4$

Hence the $x^2 + y^2 = 4$ is mapped into the circle

$(u - 3)^2 + (v - 2)^2 = 4$ in w-plane.

Problem:3

Find the image of the circle $|z| = C$ by the transformation W= $5z$.

Solu:

Given W= $5z$.

$u + iv = 5(x + iy).$

$u = 5x. \quad v = 5y$

$x = \frac{u}{5}. \quad y = \frac{v}{5}$

Given $|z| = C$

$\sqrt{x^2 + y^2} = C$

$x^2 + y^2 = c^2$

Substitute X and y values

$x^2 + y^2 = (5c)^2$

$|z| = C$ is mapped to a circle in w-plane with center at the origin and radius 5C.

Problem:4

Find the Bilinear transformation mapping the points

Z = 1, i, -1 into the points W = 2, i, -2 respectively.

Solu:

$$w_1 = 2, w_2 = i, w_3 = -2$$

$$z_1 = 1, z_2 = i, z_3 = -1$$

$$\frac{(w - w_1)}{(w - w_3)}\frac{(w_2 - w_3)}{(w_2 - w_1)} = \frac{(z - z_1)}{(z - z_3)}\frac{(z_2 - z_3)}{(z_2 - z_1)}$$

$$\frac{(w - 2)}{(w+2)}\frac{(i+2)}{(i-2)} = \frac{(z - 1)}{(Z+1)}\frac{(i+1)}{(i-1)}$$

$$\frac{(w - 2)}{(w+2)} = \frac{(z - 1)}{(Z+1)}\frac{(i+1)}{(i-1)}\frac{(i-2)}{(i+2)}$$

$$\frac{(w - 2)}{(w+2)} = \frac{(z - 1)}{(Z+1)}\frac{(3+i)}{(3-i)}$$

By using

$$\left\{ \frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d} \right.$$

$$\frac{(w-2)+(w+2)}{(w-2)-(w+2)} = \frac{(z-1)}{(Z+1)}\,\frac{(3+i)+(Z+1)(3-\mathrm{i})}{(3-\mathrm{i})-(Z+1)(3-\mathrm{i})}$$

$$\frac{-2w}{4} = \frac{6z-2i}{2iz-6}$$

$$\frac{-w}{2} = \frac{6z-2i}{2iz-6}$$

$$W = \frac{-6z+2i}{i-3}$$