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Subject Code : MA8251	**Subject Name** : ENGINEERING MATHEMATICS - II	Rev. No.	02
Unit No : 04	**Unit Name : COMPLEX INTEGRATION**	Date	14-11-2017

LECTURE NOTES

COMPLEX INTEGRATION

Introduction

Complex integration is an intuitive extension of real integration. Since a complex number represents a point on a plane while a real number is a number on the real line, the analog of a single real integral in the complex domain is always a path integral. For some special functions and domains, the integration is path independent, but this should not be taken to be the case in general. Given the sensitivity of the path taken for a given integral and its result, parameterization is often the most convenient way to evaluate such integrals. Complex variable techniques have been used in a wide variety of areas of engineering. This has been particularly true in areas such as electromagnetic field theory, fluid dynamics, aerodynamics and elasticity.

Cauchy's Theorem

Definitions

Connected Region

A connected region is one which any two points in it can be connected by a curve which lies entirely within the region.

Simply connected region

A curve which does not cross itself is called a simple closed curve. A region in which every closed curve in it encloses points of the region only is called a simply connected region.

Contour integral

An integral along a simple closed curve is called a contour integral.

Cauchy's Integral Theorem

If a function f(z) is analytic and its derivative $f_0(z)$ is continuous at all points inside and on a simple closed curve c, then $\oint_c f(z)dz = 0$

Cauchy's Integral formula

If f(z) is analytic inside and on a closed curve c of a simply connected region R and if a is any point with in c, then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z - a}dz$$

the integration around c being taken in the positive direction.

Cauchy's integral formula for derivative

If a function f(z) is analytic within and on a simple closed curve c and a is any point lying in it, then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z - a}dz$$

Worked out examples

1. Evaluate $\int_C \frac{z}{z-2} dz$ where c is a circle $|z| = 1$

   Solution:

   Let f(z) = $\frac{z}{z-2}$

   Z = 2 lies outside c.

   f(z) is analytic inside c

   f'(z) is continuous inside c

   Hence by Cauchy's theorem $\int_c f(z)dz = 0$

2. **Evaluate $\int_C \frac{1}{2z-3} dz$ where c is a circle $|z| = 1$**

   Solution:

   Given $\int_C \frac{1}{2z-3} dz = \frac{1}{2}\frac{1}{z-\frac{3}{2}} dz$ $z = \frac{3}{2}$ lies outside c.

   f(z) is analytic inside c

f'(z) is continuous inside c

Hence by Cauchy's theorem $\int_c f(z)dz = 0$

3. Evaluate $\int_C \frac{1}{2z+3} dz$ **where c is a circle** $|z| = 2$

$$\frac{1}{2z+3}dz = \int_c \frac{1}{2\left(z+\frac{3}{2}\right)dz}$$

$$= \frac{1}{2}\int_c \frac{1}{z+\frac{3}{2}dz}$$

$$= \frac{1}{2}2\pi i f\left(\frac{-3}{2}\right)$$

$$= \pi i \frac{1}{2z+3}dz$$

$$= \pi i$$

Taylor's and Laurent't Series Expansion

Taylor's Series

A function f(z), analytic inside a circle C with centre at a, can be expanded in the series

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots$$

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LECTURE NOTES

Laurent's Series:

Let C_1; C_2 be two concentric circles of radii R_1 and R_2 where $R_2 < R_1$:

Let f(z) be analytic on C_1 and C_2 and in the annular region R between them. Then, for any point in R, $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z-a)^n}$

Where

$$a_n = \frac{1}{2\pi i}\int_{c1} \frac{f(z)}{(z - a)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i}\int_{c2} \frac{f(z)}{(z - a)^{1-n}} dz$$

where the integrals being taken anticlockwise.

Problems:

1. **Expand e^z in a Taylor's series about z = 0.**

Solution:

Function	**Value at z = 0**
$f(z) = e^z$	$f(z) = 1$
$f'(z) = e^z$	$f'(z) = 1$
$f''(z) = e^z$	$f''(z) = 1$
$f'''(z) = e^z$	$f'''(z) = 1$

⋮	⋮

Taylor's series about z = 0 is

$$f(z) = f(0) + f'(0)(z) + \frac{f''(0)}{(2!)}(z)^2 + \frac{f''(0)}{2!}(z)^2 + \frac{f'''(0)}{3!}(z)^3 + \ldots$$

$$= 1 + 1(z) + \frac{1}{2!}(z)^2 + \frac{1}{3!}(z)^3 + \ldots$$

2. **Expand $\frac{1}{z-2}$ at z = 1 in a Taylor's series.**

Solution:

Function	Value at z = 1
f (z) = e^z	f(z) = 1
f'(z) = e^z	f'(z) = 1
f''(z) = e^z	f''(z) = 1
f'''(z) = e^z	f'''(z) = 1
⋮	⋮

Taylor's series about z = 1 is

$$f(z) = f(1) + \frac{f''(1)}{2!}(z)^2 + \frac{f'''(1)}{3!}(z)^3 + \ldots$$

$$= -1 + (-1)(z-1) + \frac{-2}{2!}(z-1)^2 + \frac{-6}{3!}(z-1)^3 + \ldots$$

$$= -1 - (z-1) - (z-1)^2 - (z-1)^3 - \ldots$$

2. Expand $f(z) = \frac{1}{z(z-1)}$ as a Laurent's series in powers of z and state the respective region stability

Solution:

Given $f(z) = \frac{1}{z(z-1)}$

f(z) is not analytic at z = 0 and z = 1. But it is analytic in the region

1) $0 < |z| < 1$ (deleted disc)

2) $|z| > 1$

Case (i)

For all z in $0 < |z| < 1$

$$\frac{1}{z\,(z-1)} = \frac{-1}{z\,(1-z)}$$

$$= \frac{-1}{z}[1 + z + z^2 + \ldots]$$

$$= -\left[\frac{1}{z} + z + z^2 + \ldots\right]$$

Case (ii)

For all z in $|z| > 1$ we have $\frac{1}{z} < 1$

$$\frac{1}{z(z-1)} = \frac{1}{z}\frac{1}{z-1}$$

$$= \frac{1}{z^2}\left[1 - \frac{1}{z}\right]^{-1}$$

$$= \frac{1}{z^2}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right]$$

Singularities

Definitions

Zeros of an analytic function:

If a function f(z) analytic in a region R is zero at a point $z = z_0$ in R then z_0 is called a zero of f(z).

Simple Zero:

If $f(z_0) = 0$ and $f^0(z_0) = 0$ then $z = z_0$ is called a simple zero of f(z) or a zero of the first order.

Zero of order n:

An analytic function f(z) is said to have a zero of order n if f(z) can be expressed as

$f(z) = (z-z_0)^n$ (z) where f(z) is analytic and $f^0(z_0) = 0$

Singular Points:

A point $z = z_0$ at which a function f(z) fails to be analytic is called a singular point.

Entire function

A function f(z) which is analytic everywhere in the finite plane is called an entire function.

Meromorphic function

A function f(z) which is analytic everywhere in the finite plane except at finite number of poles is called a meromorphic function.

Types of Singularities

Isolated Singularity

A point $z = z_0$ is said to be isolated singularity of f(z) if

1. f(z) is not analytic at $z = z_0$.
2. There exist a neighbourhood of $z = z_0$ containing no other singularity

Removable singularity:

If the principal part of f(z) in Laurent series expansion of f(z) about the point z_0 is zero then the point $z = z_0$ is called removable singularity.

Pole:

If we can find a positive integer n such that $\lim_{z\to a}(z-a)^n f(z) = 0$ then z = a is called a pole of order n for f(z).

Essential singularity:

If the principal part of f(z) in Laurent series expansion of f(z) about the point z_0 contains in finite number of non-zero terms then the point $z = z_0$ is called essential singularity

Problems:

1. **Find the zeros of $\frac{z^3 - 1}{z^3 + 1}$**

   Solution:

   The zeros of f(z) are given by f(z) = 0

   That is $z^3 - 1 = 0$.

   Therefore, $z = 1, \omega, \omega^2$.

2. **Find the zeros of $\frac{\sin(z) - z}{z^3}$.**

   Solution:

   The zeros of f(z) are given by f(z) = 0

   That is

$$f(z) = \frac{\sin(z) - z}{z^3}$$

$$= \frac{\frac{-z^3}{3!} + \frac{z^5}{5!} - \dots}{z^3}$$

$$= \frac{-1}{3!} + \frac{z^2}{5!} - \dots$$

Now, $\lim_{z\to 0} \frac{\sin(z) - z}{z^3} \neq 0$

3. What is the nature of the singularity z = 0 of the function $\frac{\sin(z) - z}{z^3}$

Solution:

Given f(z) = $\frac{\sin(z) - z}{z^3}$ the function f(z) is not defined at z = 0.

By L'Hospital rule

The function f(z) is not defined at z = 0

By L'Hospital rule

$$\lim_{z \to 0} \frac{\sin z = z}{z^3}$$
$$= \lim_{z \to 0} \frac{\cos z - 1}{3z^2}$$
$$= \lim_{z \to 0} \frac{-\sin z}{6z} = \frac{-1}{6}$$

Since the limit exist and is finite, the singularity at z = 0 is a removable singularity.

4. Find the nature of singularity at z = 0 of $f(z) = \frac{\sin z}{z}$

Solution:

Given that $f(z) = \frac{\sin z}{z}$ the function is not defined at z = 0.

By L'Hospital rule

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{\cos z}{1}$$

Since the limit exist and is finite, the singularity at z = 0 is a removable singularity.

Residues

Definitions:

If z = z_0 is isolated singular point of f(z), we can't find the Laurent's series of f(z) about z = z_0.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

The coefficient of $\frac{1}{z - z_0}$ in the above expression is called the residue of f(z) at z = z_0.

Problems

1. **Calculate the residue of $f(z) = \frac{1 - e^{2z}}{z^3}$**

Solution:

Given $f(z) = \frac{1 - e^{2z}}{z^3}$

Here z = 0 is a pole of order 3.

Res at (z = 0) = $\frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left[z^3 \frac{1 - e^{2z}}{z^3} \right]$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d}{dz} [-2e^{2z}]$$

$$= \frac{1}{2!} \lim_{z \to 0} [-4e^{2z}]$$

$$= -2$$

2. **Find the residue of $f(z) = \frac{z}{(z-1)^2}$ at its pole.**

Solution:

Given

$$f(z) = \frac{z}{(z - 1)^2}$$

Here z = 1 is a pole of order 2.

$$Res\ at\ (z = 1) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \left[(z - 1)^2 \frac{z}{(z - 1)^2} \right]$$

$$= \lim_{z \to 1} \frac{d}{dz} z$$

$$= 1$$

Cauchy Residue Theorem

If f(z) be analytic at all points inside and on a simple closed curve c, except for a finite number of isolated singularities z_1; z_2; z_3 then

$$\int_c f(z)\, dz = 2\pi i[\text{sum of residues}]$$

Problems

1. Let $f(z) = z^2 e^{\frac{1}{z}}$ where c is $|z| = 1$

Solution:

Here z = 0 is the only singular point which lies inside c.

$$f(z) = z^2 e^{\frac{1}{z}} = z^2\left[1 + \frac{\frac{1}{z}}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \ldots.\right]$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6}\frac{1}{z} + \ldots$$

The residue at z = 0 is $\frac{1}{6}$

By Cauchy residue theorem,

$$\int_c f(z)\, dz = 2\pi i\ (\text{sumofresidues})$$

$$\int_c f(z)\, dz = \frac{\pi i}{3}$$

2. Evaluate $\int_c \frac{e^{z^2}}{\cos\pi z}\, dz$ where c is $|z| = 1$.

Solution:

The singular points are obtained by

$$\cos \pi z = 0$$

$$\pi z = (2n + 1)\frac{\pi}{2}, n = \pm 1, \pm 2, \ldots$$

$$= \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}, \ldots.$$

$$\text{Here} z = \frac{1}{2} \text{and} z = \frac{-1}{2} \text{liesinsidec.} R_1\left[f(z), \frac{1}{2}\right] = \frac{\phi\left(\frac{1}{2}\right)}{\psi'\left(\frac{1}{2}\right)}$$

$$f(z) = \frac{e^{z^2}}{\cos\pi z} = \frac{\phi(z)}{\psi(z)}$$

$$R_1\left[f(z), \frac{1}{2}\right] = \frac{e^{\frac{1}{4}}}{-\pi}$$

$$R_2\left[f(z), \frac{-1}{2}\right] = \frac{e^{\frac{1}{4}}}{\pi}$$

By Cauchy residue theorem, $\int_c f(z)\, dz = 2\pi i(\text{sumofresidues})$

$$\int_c f(z)\, dz = 2\pi i(\text{sumofresidues}) = 0$$

Evaluation of real definite integrals on contour integrals.

Contour Integration:

The complex integration along the curve used in evaluating the definite integral is called contour integration. Here we are going to see under three types. They are

1. **Type I** – Integrals of the form
   $\int_0^{\pi} \pi\, f(\cos(\theta), \sin(\theta)) d\theta$ where f is rational function in cosθ and sinθ.
2. **Type II** – Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$.
3. **Type III** – Integrals of the form $\int_{-\infty}^{\infty} f(x)\cos(nx)\, dx \text{ or } \int_{-\infty}^{\infty} f(x)\sin(nx)\, dx$

Type I

Problems

1. **Evaluate** $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

Solution:

Let $z = e^{i\theta}$

$$dz = izd\theta$$

$$\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right] \text{ and } \sin\theta = \frac{1}{2i}\left[z - \frac{1}{z}\right]$$

$$Now, \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_c \frac{1}{2 + \frac{1}{2}\left[\frac{z^2+1}{z}\right]} \frac{dz}{iz} \text{ where } c \text{ is } |z| = 1 = \frac{2}{i}\int_c \frac{dz}{z^2+4z+1}$$

$$z^2 + 4z + 1 = 0 \Rightarrow z = -2 \pm \sqrt{3}$$

$$\alpha = -2 + \sqrt{3} \text{ is simple pole and lies inside } c$$

$$and\ \beta = -2 - \sqrt{3}\ is\ simple\ pole\ and\ lies\ outside\ c$$

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2}{i}\int_c \frac{d}{(z - \alpha)(z - \beta)}$$

$$Res|f(z), \alpha| = \lim_{z \to \alpha}(z - \alpha)\frac{1}{\left((z - \alpha)(z - \beta)\right)}$$

$$= \frac{1}{\alpha - \beta}$$

$$= \frac{1}{2\sqrt{3}}$$

2. **Using contour integration evaluate** $\int_c^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$

Solution:

$$Let\ z = e^{i\theta}$$
$$dz = iz\,d\theta$$

$$\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right]\ and\ \sin\theta = \frac{1}{2i}\left[z - \frac{1}{z}\right]$$

Now,

$$\int_0^{2\pi} \frac{d\theta}{13 + 5\sin} = \int_c \frac{1}{13 + 5\frac{1}{2i}\left[\frac{z^2 - 1}{z}\right]}\frac{dz}{iz}\ where\ c\ is\ |z| = 1$$

$$\frac{2}{5}\int_c \frac{1}{z^2 + \frac{26}{5}iz - 1}$$

$$z^2 + \frac{26}{5}iz - 1 = 0 \Rightarrow z = -5i, \frac{-i}{5}$$

$\alpha = \frac{-i}{5}$ is simple pole and lies inside c and $\beta = -5i$ *is simple pole and lies outside c*

$$\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta} = \frac{2}{5}\int_c \frac{dz}{(z - \alpha)(z - \beta)}$$

$$Res[f(z), \beta] = \lim_{z\to\infty}(z - \beta)\frac{1}{(z - \alpha)(z - \beta)}$$

$$= \frac{1}{\beta - \alpha}$$

$$= \frac{5}{24i}$$

Hence by Cauchy Residue theorem, $\int_c f(z)\, dz = 2\pi i$ (sumofresidues)

$$= \frac{5}{12}\pi \int_0^{2\pi} \frac{d\theta}{13 + 5\sin} = \frac{2}{5}\left(\frac{5}{12}\pi\right) = \frac{\pi}{6}$$

Problems on Type II:

1. **Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$ using contour integration.**

Solution:

Let us consider $\int_c f(z)\ dz = \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$

Where c consists of the semi circle Γ: $|z| = R$ *and the bounding diameter* [-R, R]

Now, $\int_c f(z)\ dz = \int_{-R}^{R} f(x)\ dx + \int_\Gamma f(z)\, dz$

The poles of f(z) are obtained by $(z^2 + 1)(z^2 + 4) = 0$

$$i.e., z = i, -i, 2i, -2i$$

Where z = 2i, I are simples lie inside Γ and z = -I, -2i pole lie outside Γ

$$R_1[f(z), i] = \lim_{z\to i}(z - i)\, f(z)$$

$$= \lim_{z\to i}(z - i)\frac{z^2}{(z + i)(z - i)(z^2 + 4)}$$

$$= \lim_{z\to i}(z - i)\frac{z^2}{(z + i)(z^2 + 4)}$$

$$= \frac{i^2}{(i + 1)(i^2 + 4)}$$

$$= -\frac{1}{6i}$$

where c consist of the semi circle : jzj = R and the bounding di-ameter [R; R]. where z = i; 2i are simple poles lie inside and z = I; 2i are simple poles lie outside.

$$R_2[f(z), 2i] = \lim_{z\to 2i}(z - 2i) f(z)$$

$$= \lim_{z\to i}(z - 2i)\frac{z^2}{(z + 2i)(z - 2i)\,(z^2 + 1)}$$

$$= \lim_{z\to i}\frac{z^2}{(z + 2i)\,(z^2 + 1)}$$

$$= \frac{(2i)^2}{(2i + 2i)((2i)^2 + 1)} = \frac{1}{3i}$$

Hence by Cauchy's Residue theorem,

$$\int_c f(z)\, dz = 2\pi i[R_1 + R_2]$$

$$= 2\pi i\left[\frac{-1}{6i} + \frac{1}{3i}\right]$$

$$= 2\pi\left(\frac{1}{6}\right) = \frac{\pi}{3}$$

$$\int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz = \frac{\pi}{3}$$

$when\ R \to \infty,$

The semicircle becomes very large and the real and imaginary parts of any point on the semi-circle becomes very large so that

$$|z| \to \infty$$

$$R \to \infty\ then \int_{\Gamma} f(z)\, dz \to 0$$

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi}{3}$$

2. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$

Solution:

Consider $\int_c f(z)\, dz = \displaystyle\int_c \frac{dz}{(z^2+1)^3}$ where c is the upper half of the semicircle Γ with the bounding diameter [R;R]

By Cauchy's Residue theorem,

$$\int_c f(z)\ dz = \int_{-R}^{R} f(x)\ dx\ +\ \int_\Gamma f(z)\, dz$$

The poles of f(z) are obtained by $(z^2\ +\ 1)^3\ =\ 0$

i.e., z = i, -i, where z = i is a pole of order 3 which lies inside *Γ* and z = -i the semicircle becomes very large and the real and imaginary parts of any point lying on the semicircle becomes very large so that, a pole of order 3 which lies outside *Γ*

$$Res[f(z), i]\ =\ \lim_{z\to i}\frac{1}{2!}\frac{d^2}{dz^2}[(z\ -\ i)^3 f(z)]$$

$$=\ \lim_{z\to i}\frac{1}{2!}\frac{d^2}{dz^2}\left[(z\ -\ i)^3\frac{1}{(z\ +\ i)^3(z\ -\ i)^3}\right]$$

$$=\lim_{z\to i}\frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{1}{(z\ +\ i)^3}\right]$$

$$=\ \lim_{z\to i}\frac{1}{2}\frac{12}{(z\ +\ i)^5}$$

$$=\frac{3}{16i}$$

$$\int_{-R}^{R} f(x)\ dx\ +\int_\Gamma f(z)\, dz = \frac{3\pi}{8}$$

When $R\ \to\infty,$ the semi-circle becomes very large and the real and imaginary parts of any point lying on the semi-circle becomes very large that $|z|\to\infty$

$$R\to\infty\ then\ \int_\Gamma f(z)\, dz\ \to 0.$$

$$\int_{-\infty}^{\infty} f(x)\ =\ \frac{3\pi}{8}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^3} dx = \frac{3\pi}{8}$$

Type III:

Problems

1. Evaluate $\int_0^{\infty} \frac{\cos(ax)}{x^2+1} dx, a > 0.$

Solution:

WKT, $\int_0^{\infty} \frac{\cos(ax)}{x^2+1} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1} dx$

To find $\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1} dx$

Consider,

$$\int_c f(z)dz = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(zx)}{z^2 + 1} dz$$

$$= R.P \int_c \frac{e^{iaz}}{z^2 + 1} dz$$

Where c is the upper half of the semi-circle Γ with the bounding diameter $[-R, R]$.

By Cauchy's residue theorem,

$$\int_c f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz$$

The poles of f(z) are obtained by $(z^2 + 1) = 0$

i.e., z = i, -i.

where z = i is a simple pole which lies inside *Γ* and z = -i is a simple pole which lies outside *Γ*

$$Res[f(z), i] = \lim_{z \to i}(z - i)$$

$$= \lim_{z \to i}(z - i)\frac{e^{iaz}}{z^2 + 1}$$

$$= \lim_{z \to i}\frac{(z - i)(ia)e^{iaz} + e^{iaz}}{2z}$$

$$= \frac{e^{-a}}{2i}$$

Hence by Cauchy's Residue theorem,

$$\int_c f(z)\, dz = R.P\ 2\pi i[\text{sum of residues}]$$

$$= R.P\ 2\pi i\frac{e^{-a}}{2i}$$

$$= R.P\ \pi e^{-a}$$

$$\int_{-R}^{R} f(x)\ dx + \int_{\Gamma} f(z)\, dz = \pi e^{-a}$$

$$\text{If } R \to \infty, \text{then} \int_{\Gamma} f(z) dz \to 0$$

$$\int_{-\infty}^{\infty} f(x)\, dx = \pi e^{-a}$$

$$\int_0^\infty \frac{\cos(ax)}{x^2 + 1} dx = \frac{\pi}{2} e^{-a}$$

2. Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

Solution:

WKT

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{1}{2} \int_\infty^\infty \frac{\sin x}{x} dx$$

To find $\int_\infty^\infty \frac{\sin x}{x} dx$ consider

$$\int_c f(z)\, dz = \int_c \frac{\sin(z)}{z} dz$$

$$= \text{I.P} \int_c \frac{e^{iz}}{z} dz$$

Where c is the upper half of the semi-circle Γ with the bounding diameter [-R, R]

i.e., $\int_{-R}^{R} f(x)\ dx + \int_\Gamma f(z)\, dz$

The poles of f(z) are obtained by z = 0

Z = 0 is a simple pole lies on the real axis inside Γ

$$\text{Res}[f(z), 0] = \lim_{z \to 0} (z) f(z)$$

$$= \lim_{z\to 0}(z)\frac{e^{iz}}{z}$$

$$= \lim_{z\to 0} e^{iz} = 1$$

Hence, by Residue Theorem,

$$\int_c f(z)\, dz = 2\pi i(0) + \pi i(1)$$

$$\text{I.P}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx = \text{I.P}\ \pi i$$

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \pi$$

And therefore, $\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$

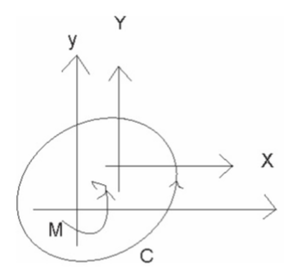
Applications:

Blasius Theorem.

The following figure shows a cross-section of a cylinder (not necessarily cir-cular), whose boundary is C,placed in a steady non-viscous flow of an ideal fluid; the flow takes place in planes parallel to the xy plane. The cylinder is out of the plane of the paper. The flow of the fluid exerts forces and turning moments upon the cylinder. Let X, Y be the components, in the x and y directions respectively, of the force on the cylinder and let M be the anticlockwise moment (on the cylinder) about the origin.

$$w = U\left(z + \frac{a^2}{z}\right)$$

so that $\frac{dw}{dz} = U\left(1 - \frac{a^2}{z}\right)$



Blasius Theorem

$$\left(\frac{dw}{dz}\right)^2 = U^2\left(1 - \frac{2a^2}{z^2} + \frac{a^4}{z^4}\right)$$

Now, $X - iY = \frac{1}{2}i\rho\int_C\left(\frac{dw}{dz}\right)^2 dz = \frac{1}{2}i\rho U^2\int_C\left(1 - \frac{2a^2}{z^2} + \frac{a^4}{z^4}\right)dz = 0$

Hence,

X = Y = 0.

Also, $z\left(\frac{dw}{dz}\right)^2 = U^2\left(\left(1 - \frac{2a^2}{z^2} + \frac{a^4}{z^4}\right)\right)$

The only term to contribute to M is $\frac{-2a^2U^2}{z}$.

Again using the Key Point above this leads to 4 a^2U^2i and this has zero real part. Hence M = 0, also. The implication is that no net force or moment acts on the cylinder. This is not so in practice. The discrepancy arises from neglecting the viscosity of the fluid.