Course/Branch : MA8251/Common to all	**Year / Semester** : I/02	Format No.	NAC/TLP-07a.5
Subject Code : MA8251	**Subject Name** : ENGINEERING MATHEMATICS - II	Rev. No.	02
Unit No : 05	**Unit Name : LAPLACE TRANSFORMS**	Date	14-11-2017

LECTURE NOTES

LAPLACE TRANSFORMATION

- Laplace transformation – Conditions and existence
- Transforms of elementary functions – Basic properties
- Transforms of derivatives, Derivatives and Integrals of Transforms, Integrals of transforms
- Transforms of the unit step functions and impulse function
- Transforms of periodic functions
- Inverse Laplace Transforms
- Convolution Theorem
- Initial and Final Value Theorem
- Solution of linear ODE of second order with constant coefficients

Laplace Transform

Definitions:

Transformation:

A "Transformation" is an operation which converts a mathematical expression to a different but equivalent form

Laplace transformation:

Let a function f(t) be continuous and defined for positive values of 't'. The Laplace Transformation of f(t) associates a function s defined by equation

$$L[f(t)] = \int_0^\infty e^{-st} f(t), t > 0$$

Here, F(s) is said to be the Laplace transform of f(t) and it is written as L[f(t)] or L[f]. Thus,

F(s) = L[f(t)]

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt, t > 0$$

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LECTURE NOTES

Exponential order:

A function f(t) is said to be of exponential order if

$$\lim_{t\to\infty} e^{-st} f(t) = 0$$

Laplace Transform – sufficient conditions for existence

- F(t) should be continuous or piecewise continuous in the given closed interval [a, b] where a>0
- F(t) should be of exponential order

Example

1. L[tan t] does not exist since tan t is not piecewise continuous i.e., tan t has infinite number of infinite

$$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Problems:

Given $f(t) = t^2$

By the definition of exponential order,

$$\lim_{t\to\infty} e^{-st} f(t) = \lim_{t\to\infty} e^{-st} t^2$$

$$= \lim_{t\to\infty} \frac{t^2}{e^{st}} \left[\frac{\infty}{\infty}\ Indefinite\ form, Apply\ L'Hospital\ rule\right]$$

$$= \lim_{t\to\infty} \frac{2t}{se^{st}} \left[\frac{\infty}{\infty}\ Indefinite\ form, Apply\ L'Hospital\ rule\right]$$

$$= \lim_{t\to\infty} \frac{2}{s^2 e^{st}}$$

$$= \frac{2}{\infty}$$

$$\lim_{t\to\infty} e^{-st} t^2 = 0$$

Here t^2 is of exponential order.

2. Show that the function given is not of exponential order $f(t) = e^{t^2}$

Solution:

Given

$$f(t) = e^{t^2}$$

By the definition of exponential order,

$$\lim_{n\to\infty} e^{-st} e^{t^2} = \lim_{n\to\infty} e^{-st + t^2}$$

$$= e^{\infty}$$

$$\lim_{n\to\infty} e^{-st} e^{t^2} = \infty$$

Definition function of class A

A function which is sectionally continuous over any finite interval and is of exponential order is known as a function of class A

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LECTURE NOTES

Transforms of elementary functions – Basic properties

Important Results

S.No.	Laplace transform pair	Conditions
1	$L[1] = \frac{1}{s}$	where s>0
2	$L[t^n] = \frac{n!}{s^{n-1}}$	where n = 0, 1, 2,…
3	$L[t^n] = \frac{\Gamma n + 1}{s^{n+1}}$	where n is not a integer
4	$L[e^{at}] = \frac{1}{s - a}$	where s > a or s – a > 0
5	$L[e^{-at}] = \frac{1}{s + a}$	where s + a > 0
6	$L[\sin at] = \frac{a}{s^2 + a^2}$	where s > 0
7	$L[\cos at] = \frac{s}{s^2 + a^2}$	where s > 0
8	$L[\sinh at] = \frac{a}{s^2 - a^2}$	where $s > \lvert a \rvert$ or $s^2 > a^2$
9	$L[\cosh at] = \frac{s}{s^2 + a^2}$	where $s^2 > a^2$
10	Linearity property $L[af(t) \pm bg(t)] = aL[f(t)] \pm bL[g(t)]$	

Problems

3. Find $L[t^5 + e^{3t} + e^{-2t}]$

Solution:

$$L[t^5 + e^{3t} + e^{-2t}] = L[t^5] + L[e^{3t}] + L[5e^{-2t}]$$

$$= \frac{5!}{s^{s+1s+2}} + \frac{1}{s - 3} + \frac{5}{s + 2}$$

$$L[t^5 + e^{3t} + e^{-2t}] = \frac{5!}{s^6} + \frac{1}{s - 3} + \frac{5}{s + 2}$$

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LECTURE NOTES

4. Find $L\left[\frac{1}{\sqrt{t}}\right]$

Solution:

Given $L\left[\frac{1}{\sqrt{t}}\right] = L\left[t^{\frac{-1}{2}}\right]$

$$= \Gamma\left(\frac{-1}{2} + 1\right)$$

$$= \frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

Hence, $L\left[\frac{1}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}}$

First Shifting Theorem

If $L[f(t)] = F(s), then\ L[e^{at}f(t)] = F(s - a)$

If $L[f(t)] = F(s), then\ L[e^{-at}f(t)] = F(s + a)$

Second Shifting Theorem

If $L[f(t)] = F(s)$and

G (t) $\begin{cases} = f(t - a), t > a \\ 0, t < a \end{cases}$

Then $L[f(t)] = e^{-as}F(s)$

Problems on First and Second Shifting Theorem

1. Find $L[t^n e^{-at}]$

Solution:

$$L[t^n e^{-at}] = \left[L[t^n]\right]_{s \to (s+a)}$$

$$= \left[\frac{n!}{s^{n+1}}\right]_{s \to (s+a)}$$

$$L[t^n e^{-at}] = \left[\frac{n!}{(s+a)^{n+1}}\right]$$

2. Find $L[e^{at}\sinh bt]$

Solution:

$$L[e^{at}\sinh bt] = \left[L[e^{at}\sinh bt]\right]_{s \to (s-a)}$$

$$= \left[\frac{b}{s^2 - b^2}\right]_{s \to (s-a)}$$

$$L[e^{at}\sinh bt] = \frac{b}{(s-a)^2 - b^2}$$

Transforms of derivatives and Integrals of Functions

Properties:

- $L[f'(t)] = sL[f(t)] - f(0)$
- $L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0)$

Transform of integrals:

If $L[f(t)] = F(s)$,then

$$L\left[\int_0^t f(u)\ du\right] = \frac{1}{s}L[f(t)]$$

Derivatives of transform

- If $L[f(t)] = F(s)\ then\ L[t\ f(t)] = -\frac{d}{ds}F(s) = -F'(s)$
- If $L[f(t)] = F(s)\ then\ L[t^n f(t)] = (-1)^n F^{(n)}(s)$

1. Find $L[t \sin at]$

Solution:

We know that

$$L[f(t)] = F(s)\ then\ L[t\ f(t)] = -\frac{d}{ds}F(s) = (-1)^n F'(s)$$

$$L[t \sin at] = -\frac{d}{ds}L[\sin at]$$

$$= -\frac{d}{ds}\left[\frac{a}{s^2 + a^2}\right]$$

$$= \left(\frac{(s^2 + a^2)(0) - a(2s)}{(s^2 + a^2)^2}\right)$$

$$= -\left(\frac{-a(2s)}{(s^2 + a^2)^2}\right)$$

$$L[t \sin at] = \frac{2as}{(s^2 + a^2)^2}$$

2. Show that $\int_0^\infty e^{-t} \cos t \, dt = 0$

Solution:

Given $\int_0^\infty e^{-t} \cos t \, dt = \left[L[t\cos t]\right]_{s\,=\,1}$

$$= \left[-\frac{d}{ds} L(\cos t)\right]_{s\,=\,1}$$

$$= \left[-\frac{d}{ds} L\left(\frac{s}{s^2 + 1}\right)\right]_{s\,=\,1}$$

$$= \left[-\left(\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right)\right]_{s\,=\,1}$$

$$= \left[-\left(\frac{s^2 + 1 - 2s^2}{(s^2 + 1)^2}\right)\right]_{s\,=\,1}$$

$$= \left[-\left(\frac{s^2 + 1 - 2s^2}{(s^2 + 1)^2}\right)\right]_{s\,=\,1} = \left[-\left(\frac{1 - s^2}{(s^2 + 1)^2}\right)\right] = [-0]$$

Therefore $\int_0^\infty e^{-t} \cos t \, dt = 0$

Problems based on Integral of Transform

1. Find $L\left[\frac{\sin 3t \cos t}{t}\right]$

   Solution:

   Given $L\left[\frac{\sin 3t \cos t}{t}\right] = L\left[\frac{\sin 4t + \sin 2t}{2t}\right]$

   $= \frac{1}{2} L\left[\frac{\sin 4t + \sin 2t}{2t}\right]$

$$= L\left[\frac{\sin 3t \cos t}{t}\right] = \frac{1}{2}\left[L\left[\frac{\sin 4t}{t}\right] + L\left[\frac{\sin 2t}{t}\right]\right]$$

$$= \frac{1}{2}\left[cot^{-1}\left(\frac{s}{4}\right) + cot^{-1}\left(\frac{s}{2}\right)\right] \text{since} \left(L\left[\frac{\sin at}{t}\right] = cot^{-1}\left(\frac{s}{a}\right)\right)$$

Therefore, $L\left[\frac{\sin 3t \cos t}{t}\right] = = \frac{1}{2}\left[cot^{-1}\left(\frac{s}{4}\right) + cot^{-1}\left(\frac{s}{2}\right)\right]$

Transforms of Unit Step Function and Impulse Function:

Problems based on unit step function or Heaviside's Unit step function

1. **Define the unit step function**
   Solution:
   The Unit step function, also called Heaviside's unit function is defined as
   $U(t - a) = \begin{cases} 0 \; for \; t < a \\ 1 \; for \; t > a \end{cases}$
   This is the unit step function at t = a. It can also be denoted by H(t – a).
2. Give the L.T of the unit step function
   Solution:

$$L[U(t - a)] = \int_0^\infty e^{-at}\, U(t - a)\, dt$$

$$= \int_0^a e^{-st}\,(0)\, dt + \int_0^\infty e^{-st}\,(1)\, dt$$

$$= \int_0^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_s^\infty$$

$$= 0 - \left[\frac{e^{-sa}}{-s}\right]$$

$$L[U(t - a)] = = \left[\frac{e^{-sa}}{s}\right]$$

Transform of Periodic Functions

Definition: (Periodic)

A function f(x) is said to be "periodic" if and only if f(x + p) = f(x) is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace Transformation of a periodic function f(t) with period p given by
$\frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)dt$

Problems:

1. Find the Laplace Transform of the Half-sine wave rectifier function

$$f(t) = \begin{cases} \sin \omega t, 0 < t < \frac{\pi}{\omega} \\ 0, \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

Solution:

We know that

$$L[f(t)] = \frac{1}{1 - e^{-ps}}\int_{0\,dt}^{\frac{2\pi}{\omega}} e^{-st} f(t)\, dt$$

$$L[\sin \omega t] = \frac{1}{1 - e^{-ps}}\left(\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\, dt + 0\right)$$

$$L[\sin \omega t] = \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}}\left(\frac{e^{-st}}{s^2 + \omega^2}[-s \sin \omega t - \omega \cos \omega t]\right)_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}}\left(\frac{e^{-s\pi}\omega + \omega}{s^2 + \omega^2}\right)$$

$$= \frac{\omega(1 + e^{-s\pi})}{\left(1 - e^{\frac{-s\pi}{\omega}}\right)\left(1 + e^{\frac{-s\pi}{\omega}}\right)(s^2 + \omega^2)}$$

$$= \frac{\omega}{\left(1 - e^{\frac{-s\pi}{\omega}}\right)(s^2 + \omega^2)}$$

$$L[\sin \omega t] == \frac{\omega}{(s^2 + \omega^2)\left(1 - e^{\frac{-s\pi}{\omega}}\right)}$$

Inverse Laplace Transform

a. If L[f(t)] = F(s), then
   $L^{-1}[F(s)] = f(t)\ where\ L^{-1}$ is called the inverse Laplace transform operator.
b. If $F_1(s)\ and\ F_2(s)\ are\ L.T. of\ f(t)\ and\ g(t)$ respectively then
   $L^{-1}[C_1F_1(s) + C_2F_2(s)] = C_1L^{-1}[F_1(s)] + C_2[F_2(s)]$

<u>Problems on inverse Laplace Transform</u>

1. Find $L^{-1}\left[\frac{2s}{s^2 - 16}\right]$

   <u>Solution:</u>

   $L^{-1}\left[\frac{2s}{s^2 - 16}\right] = 2L^{-1}\left[\frac{s}{s^2 - 16}\right]$

   $= 2\cosh 4t$

2. Find $L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]$

   <u>Solution:</u>

$$L^{-1}\left[\frac{s - 3}{s^2 + 4s + 13}\right] = L^{-1}\left[\frac{s - 3}{(s + 2)^2 + 13 - 4}\right]$$

$$= L^{-1}\left[\frac{s - 3}{(s + 2)^2 + 9}\right]$$

$$= L^{-1}\left[\frac{s+2-5}{(s+2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - 5L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] - 5L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{s}{(s+2)^2+3^2}\right] - \frac{5}{3}L^{-1}\left[\frac{3}{(s+2)^2+3^2}\right]$$

$$= e^{-2t}\cos 3t - \frac{5}{3}L^{-1}\left[\frac{3}{s^2+3^2}\right]$$

$$L^{-1}\left[\frac{s-3}{s^2+4s+13}\right] = e^{-2t}\cos 3t - \frac{5}{3}e - 2t\sin 3t$$

Inverse Laplace Transforms of derivatives of F(s)

If $L^{-1}[F(s)] = f(t), then\ L^{-1}[F'(s)] = -t\, f(t)$

$$= -tL^{-1}[F(s)]$$

Problems:

1. Find $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]$

Solution:

Let $F'(s) = \left[\frac{s}{(s^2-a^2)^2}\right]$

$$\int F'(s)ds = \int \left[\frac{s}{(s^2-a^2)^2}\right]ds$$

$$F(s) = \int \left[\frac{s}{(s^2-a^2)^2}\right]ds$$

Put $s^2 - a^2 = t$

2s ds = dt

$$s\,ds = \frac{dt}{2}$$

$$= \int \frac{1}{t^2}\frac{dt}{2} = \frac{1}{2}\left[\frac{-1}{t}\right]$$

$$= -\frac{1}{2t}$$

$$F(s) = -\frac{1}{2(s^2 - a^2)}$$

Inverse Laplace Transform of Integrals:

$$L^{-1}\left[\int_0^{\infty} F(s)\,ds\right] = \frac{1}{t}f(t) = \frac{1}{t}L^{-1}[F(s)]$$

Or $L^{-1}[F(s)] = tL^{-1}\left[\int_0^{\infty} F(s)\,ds\right]$

Problems:

1.Find $L^{-1}[F(s)] = tL^{-1}\left[\int_s^{\infty} F(s)\,ds\right]$

$$L^{-1}[F(s)] = tL^{-1}\left[\int_s^{\infty} F(s)\,ds\right]$$

$$= t\,L^{-1}\left[\left(\frac{-1}{s^2 - 1}\right)_s^{\infty}\right]$$

$$= t\,L^{-1}\left[0 + \frac{1}{s^2 - 1}\right]$$

$$= t\,L^{-1}\left[\frac{1}{s^2 - 1}\right]$$

$$L^{-1}\left[\frac{2s}{(s^2 - 1)^2}\right] = t \sinh t$$

Problems based on partial fractions method

1. Find $L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$

Solution:

Consider

$$\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3}$$

$$5s^2 - 15s - 11 = A(s-2)^3 + B(s+1)(s-2)^3 + C(s+1)(s-2) + D(s+1)$$

Put s = -1, we get

$$5 + 15 \pm 11 = A(-1-2)^3$$

$$9 = -27A$$

$$A = \frac{-1}{3}$$

Equating the coefficients of s^3 on both sides, we get

$$0 = A + B$$

B = -A

$$B = \frac{1}{3}$$

Put s = 2, we get

$$-21 = D$$

$$D = -7$$

Put s = 0, we get

$$-11 = -8A + 4B - 2C + D$$

$$= -8\left(\frac{-1}{3}\right) + 4\left(\frac{1}{3}\right) - 2C - 7$$

$$-4 = \frac{8}{3} + \frac{4}{3} - 2C$$

$$-8 = -2C$$

$$C = 4$$

$$\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} = \frac{\frac{-1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right]$$

$$= \frac{-1}{3}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3}L^{-}\left[\frac{1}{s-2}\right]$$

$$+ 4L^{-}\left[\frac{1}{(s-2)^2} - 7L^{-}\left[\frac{1}{(s-2)^3}\right]\right]$$

$$L^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right] = \frac{-1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right] - 7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$= \frac{-1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}t - 7e^{2t}L^{-1}\left[\frac{2}{s^3}\right]$$

Therefore, $L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right] = \frac{-1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}t - \frac{7}{2}e^{2t}t^2$

Second Shifting property:

$$L^{-1}[e^{-as}F(s)] = f(t-a)U(t-a)$$

Problems based on second shifting property:

1. Find $L^{-1}\left[\frac{e^{-\pi s}}{s+3}\right]$

   Solution:

   Consider

$$L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$$

$$L^{-1}\left[\frac{e^{-\pi s}}{s + 3}\right] = e^{-3(t - \pi)}U(t - \pi)$$

Change of scale property

If $L[f(t)] = F(s), then\ L[f(at)] == \frac{1}{a}F\left[\frac{s}{a}\right]$

$$f(t) = L^{-1}[F(s)], then\ L[f(cs)] == \frac{1}{c}f\left[\frac{t}{c}\right]$$

Problems based on change of scale property

1. If $L[f(t)] = F(s)\ find\ L\left[f\left(\frac{t}{a}\right)\right]$

Solution:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

We know that

$$L\left[f\left(\frac{t}{a}\right)\right] = \int_0^\infty e^{-st} f\left(\frac{t}{a}\right) dt$$

$$Put\ u = \frac{t}{a} \quad as\ t \to 0 \Rightarrow u \to 0$$

$$du = \frac{dt}{a} \quad t \to \infty \Rightarrow u \to \infty$$

$$L\left[f\left(\frac{t}{a}\right)\right] = \int_0^\infty e^{-s(au)} f(u)adu$$

$$= \int_0^\infty e^{-sat} f(t)dt$$

$$= sF[as]$$

Convolution Theorem

If f(t) and g(t) are functions defined for $t \geq 0$,

Then $L[f(t) * g(t)] = L[f(t)] \quad L[g(t)]$

Problems on Convolution:

Theorem 1. Define convolution

The convolution of two functions f(t) and g(t) is defined as

$$f(t) * g(t) = \int_0^t f(u)\, g(t - u)\, du$$

Note: Convolution Integral or Falting Integral

1. Using convolution theorem find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$

Solution:

We know that $L^{-1}[F(s).G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$

$$L^{-1}\left[\frac{1}{s+a}.\frac{1}{s+b}\right] = L^{-1}\left[\frac{1}{s+a}\right] * L^{-1}\left[\frac{1}{s+b}\right]$$

$$= e^{-at} * e^{-bt}$$

Here,

$$f(t) = e^{-at}$$

$$g(t) = e^{-bt}$$

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u) du$$

$$= \int_0^t e^{-au}\, e^{-b^{(t-u)}} du$$

$$= \int_0^t e^{-au}\, e^{-bt} e^{-bu} du$$

$$= e^{-bt}\int_0^t e^{-(a-b)u}\,du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t$$

$$= \frac{e^{-bt}}{a-b}\left[\frac{e^{-(a-b)t}}{-(a-b)} - \frac{1}{-(a-b)}\right]$$

$$= \frac{e^{-bt}}{a-b}[1 - e^{-at}e^{bt}]$$

$$L^{-1}\left[\frac{1}{s+a}.\frac{1}{s+b}\right] = \frac{1}{a-b}[e^{-bt} - e^{-at}]$$

2. Using convolution theorem find

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right]$$

Solution:

We know that

$$L^{-1}[F(s).G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 1 * \sin t$$

$$= \sin t * 1$$

$$= \int_0^t \sin\, u du$$

$$= [-\cos u]_{u=0}^{u=t}$$

$$= (-\cos t) - (-1)$$

$$L^{-1}\left[\frac{1}{s(s^2 + 1)}\right] = 1 - \cos t$$

Initial and final value theorems

Initial value theorem

$$\text{If } L[f(t)] = F(s), then \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Final value theorem

$$\text{If } L[f(t)] = F(s), then \lim_{t\to \infty} f(t) = \lim_{s\to 0} sF(s)$$

Problem based on initial and final value theorem

1. If $L[f(t)] = \frac{1}{s(s+a)}, find \lim_{t\to\infty} f(t)\ and \lim_{t\to 0} f(t)$

Solution:

We know that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$= \lim_{s\to\infty} s\frac{1}{s(s + a)}$$

$$= \lim_{s\to\infty} \frac{1}{(s + a)}$$

$$= \frac{1}{a}$$

$$\lim_{t\to 0} f(t) = \frac{1}{\infty}$$

$$\lim_{t\to 0} f(t) = 0$$

We know that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} s \frac{1}{s(s + a)}$$

$$= \lim_{s\to 0} \frac{1}{(s + a)}$$

$$\lim_{t\to\infty} f(t) = \frac{1}{a}$$

2. Verify the initial and final value theorem for the function
$$f(t) = 1 + e^{-t}(\sin t + \cos t)$$

Solution:

Initial value theorem states that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$L[f(t)] = F(s) = \frac{1}{s} + L[\sin t + \cos t]_{s\to 1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

$$L.H.S = \lim_{t\to 0} f(t) = 1 + 1 = 2$$

$$R.H.S = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty} s\left[1+\frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= \lim_{s\to\infty} s\left[1+\frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right]$$

$$= \lim_{s\to\infty}\left[1+\frac{\left(1+\frac{2}{s}\right)}{\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right]$$

$$= 1+1$$

R.H.S=2

Therefore, L.H.S=R.H.S

Initial Value theorem verified

Final value theorem states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$L.H.S = \lim_{t\to\infty}[1+e^{-t}(\sin t+\cos t)]$$

$$= 1+0 = 1$$

$$R.H.S = \lim_{s\to 0}\left[1+\frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= 1+0 = 1$$

$$L.H.S = R.H.S$$

Final value theorem verified

Problems based on solution of linear ODE of second order with constant coefficients

Course/Branch : MA8251/Common to all	**Year / Semester** : I/02	Format No.	NAC/TLP-07a.5
Subject Code : MA8251	**Subject Name** : ENGINEERING MATHEMATICS - II	Rev. No.	02
Unit No : 05	**Unit Name** : **LAPLACE TRANSFORMS**	Date	14-11-2017

LECTURE NOTES

1. Using L.T solve $y'' - 3y' + 2y = e^{-t}$ $given\ y(0) = 1, y'(0) = 0$

   Solution:

   Taking L.T on both sides

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L[e^{-t}]$$

$$s^2L[y(t)] - sy(0) - y'(0) - 3\left[sL[y'(t)] - y[0]\right] + 2L[y(t)] = \frac{1}{s+1}$$

$$s^2L[y(t)] - s - 0 - 3sL[y(t)]\ +\ 3 +\ 2L[y(t)] = \frac{1}{s+1}$$

$$(s^2 - 3s + 2)L[y(t)] = \frac{1}{s+3} + s - 3$$

$$(s-1)(s-2)L[y(t)] = \frac{s^2 - 2s - 2}{s+1}$$

$$L[y(t)] = \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$s^2 - 2s - 2 = A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)$$

$$Put\ s = 1, we\ get$$

$$1 - 2 - 2 = -2B$$

$$-3 = -2B$$

$$B = \frac{3}{2}$$

$$Put\ s = 2, we\ get$$

$$4 - 4 - 2 = 3C$$

$$C = \frac{-2}{3}$$

$$Put\ s = -1, we\ get$$

$$1 + 2 - 2 = 6A$$

$$A = \frac{1}{6}$$

$$L[y(t)] = \frac{\frac{1}{6}}{s+1} + \frac{\frac{3}{2}}{s-1} + \frac{\frac{2}{3}}{s-2}$$

$$L[y(t)] = \frac{1}{6}\frac{1}{s+1} + \frac{3}{2}\frac{1}{s-1} + \frac{2}{3}\frac{1}{s-2}$$

$$y(t) = \frac{1}{6}L^{-1}\left[\frac{1}{s+1}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right] + \frac{2}{3}\frac{1}{6}L^{-1}\left[\frac{1}{s-2}\right]$$

$$= \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} + \frac{2}{3}e^{2t}$$